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Graph Polynomials for Counting of Cycles and Paths

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Graph Polynomials for Counting of Cycles and Paths

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Abstract

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IV. Preface

This thesis is written as a completion of the bachelor education in applied mathematics from Hochschule Mittweida. I have pondered over this topic for two years and gathered a number of ideas during the studies. I would like to acknowledge everyone for their help on my study and life.
First of all, I am grateful to Professor Peter Tittmann for motivating me to do research on graph polynomials and discrete mathematics as well as for his excellent guidance, supervision and assistance. I am also thankful to my parents for their support, for enabling me to study mathematics. I thank my friends to give me a lot of ideas for study and life. In particular, I would like express my gratitude to my girlfriend Wenting Li for her understanding and moral support. Finally, I thank the Almighty Lord for the creation of mathematics full of beauty and joy.
I hope you enjoy your reading.

Xiangying Chen
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1 Introduction

Counting is one of the earliest problems in mathematics. Humans have a counting history for thousands of years, but the main principle of counting did not change. We always establish a bijection from the given set to a known set, which can be the set of fingers on two hands or an abstract structure with many algebraic properties. Graphs are an important mathematical structure, since network-like structures can be modeled as graphs. Counting cycles and paths in graphs and digraphs is a classic problem in graph theory and enumerative combinatorics. It has been well researched for over a half century. Numerous results have been obtained through using a variety of methods, for example, the adjacency matrix [8, 23, 28, 29, 31], immanantal polynomials [11], Hopf algebras [20] and many algorithms [2, 7, 9]. Since counting Hamiltonian cycles, cycles on a vertex and i-j paths are #P-complete [35], a polynomial-time algorithm is not expected.

Many graph polynomials have been introduced and well studied over the years. Several graph polynomials are generating functions for subgraphs with certain properties, for example the independence polynomial [22] enumerates edgeless induced subgraphs, the clique polynomial [25] enumerates complete induced subgraphs, the edge cover polynomial [1] enumerates spanning subgraphs without isolated vertices, the matching polynomial [16] enumerates spanning subgraphs without vertices of degree greater than 1. Subgraphs with several properties are counted by introducing new parameters, for example the Tutte polynomial [33] counts spanning subgraphs of each rank and nullity, and the subgraph component polynomial [32] counts induced subgraphs of each number of vertices and components. Graph polynomials are helpful for encoding, classifying and researching graph invariants. In this thesis, several new polynomials for enumerating cycles and paths in graphs and digraphs are introduced.

1.1 Organization of the Thesis

This thesis is structured as follows.

In Chapter 2, polynomials counting cycle subgraphs, path subgraphs, u-paths, u-v paths and subgraphs consisting of cycles are introduced. We present the properties, different representations, recurrence relations with respect to edge and vertex operations, closed-form expressions for special graph classes and distinguish power of these polynomials and relationships among them.

In Chapter 3, digraph polynomials are considered. In Section 3.1, we introduce some digraph polynomials counting directed cycles and paths. These digraph polynomials satisfy arc deletion-contraction-extraction recurrence relations like the edge elimination polynomial [5, 6] and vertex deletion-contraction recurrence relations. We give the relationships to their undirected versions and among them. In Section 3.2, we present some facts about the cover polynomial and the geometric cover polynomial, which motivate the research on the digraph polynomials. In Section 3.3, we generalize the digraph polynomials counting cycles and paths and the geometric cover polynomial to the trivariate cycle-path polynomial. In Section 3.4, applying the ideas of [5], the arc elimination polynomial is introduced, which is the most universal digraph polynomial satisfying linear recurrence relation with respect to deletion, contraction and extraction of arcs. We show that the arc elimination polynomial is co-reducible to the trivariate cycle-path polynomial. An explicit form of the arc elimination polynomial is given.

In [10], the cycle polynomial and the bivariate cycle polynomial are also introduced, and the edge decomposition formulae are stated. It is published after the completion of this thesis.

1.2 Definitions

Definition 1.1 A *simple graph* G is an ordered pair (V,E) consisting of a set $V=V(G)$ of *vertices* and a set $E=E(G)$ of *edges*, where $E\subseteq\binom{V}{2}$. In a *multigraph*, E is a multiset over a subset of $\binom{V}{2}\cup\{\{v\}\}$, that is, a ground set together with a multiplicity function $m:\binom{V}{2}\cup\{\{v\}\}\to\mathbb{N}$. An edge $e=\{v\}\in E(G),v\in V(G)$ is called a *loop*. If $e=\{v,w\}$, v and w are called *end vertices* or *ends* of $e\in E(G)$, and e is *incident* to v and w. If e is a loop, the ends are identical. Two vertices $v,w\in V(G)$ are *adjacent* if $v,w\in E(G)$. Two edges $e,f\in E(G)$ are *incident* if $e\cap f\neq\emptyset$. In a multigraph, two edges are *parallel* if they have the same end vertices.

Definition 1.2 Let $G=(V,E)$ be a (multi-)graph, $N(v):=\{w\in V(G)|\{v,w\}\in E(G)\}$ is called the *open neighborhood* of v. The *degree* $\deg(v)$ of v is the number of edges incident to v in G. An *isolated vertex* is a vertex with degree 0.

Definition 1.3 A *simple directed graph* (*simple digraph*) G is an ordered pair (V,E) consisting of a set $V=V(G)$ of *vertices* and a set $E=E(G)$ of *arcs*, where $E\subseteq V\times V$. In a *multidigraph*, the arc set E is a multiset. Two arcs are *incident* if they contain a common vertex. For an arc $(u,v)\in E$, u is said to be the *head* of (u,v) and v its *tail*, and u,v are said to be *incident* to the arc (u,v). An arc $(v,v)\in E$ is called a *loop*. Two arcs having the same head and same tail are called *parallel*, and two arcs $(u,v),(v,u)\in E$ are called *antiparallel*.

Throughout this thesis, the graphs and digraphs will be multigraphs and multidigraphs unless otherwise stated.

Definition 1.4 Let G and H be any two simple graphs, a *homomorphism* of G to H is a mapping $f:V(G)\to V(H)$ such that $\{f(u),f(v)\}\in E(H)$ if $\{u,v\}\in E(G)$. If G and H are multigraphs with loops, a homomorphism of G to H is a function $f_V:V(G)\to V(H)$ together with an associated function $f_E:E(G)\to E(H)$ consistent with f_V in that $f_E(\{u,v\})=\{f_V(u),f_V(v)\}$. An *isomorphism* is a bijective homomorphism whose inverse is also a homomorphism. A graph G is called *isomorphic* to a graph H and denoted as $G \cong H$ if there is an isomorphism of G to H.

Definition 1.5 Let $G = (V,E)$ be a graph or a digraph. A (di-)graph (W,F) is a *subgraph* of G if $W \subseteq V$ and $F \subseteq E$.
The *spanning subgraph* $G\langle F\rangle$ of (di-)graph G is the subgraph $G\langle F\rangle = (V,F)$.
The *induced subgraph* $G[W]$, $W \subseteq V$ is the subgraph with vertex set W and all edges (arcs) of G whose both ends (head and tail) are in W.

From the definitions we know, H is a subgraph of G iff $V(H) \subseteq V(G)$ and there is an injective homomorphism from H to G. H is a spanning subgraph of G iff $V(H) = V(G)$ and there is an injective homomorphism from H to G.

Definition 1.6 The *empty (di-)graph E_n* is the (di-)graph $E_n = (V, \emptyset)$, $n = |V|$.
The *complete graph K_n* is the graph $K_n = (V, \binom{n}{2})$, $n = |V|$.
The *path graph P_n* is the graph $P_n = (\{1,\dots,n\},\{\{1,2\},\{2,3\},\dots,\{n-1,n\}\})$, $n \geq 2$.
The *cycle graph C_n* is the graph $C_n = (\{1,\dots,n\},\{\{1,2\},\{2,3\},\dots,\{n-1,n\},\{n,1\})$, $n \geq 2$. C_1 is the graph with one vertex and one loop on it.
The *directed path graph $\overrightarrow{P_n}$* is the digraph $\overrightarrow{P_n} = (\{1,\dots,n\},\{(1,2),(2,3),\dots,(n-1,n)\})$, $n \geq 2$.
The *directed cycle graph $\overrightarrow{C_n}$* is the digraph $\overrightarrow{C_n} = (\{1,\dots,n\},\{(1,2),(2,3),\dots,(n-1,n),(n,1)\})$, $n \geq 2$. $\overrightarrow{C_n}$ is the digraph with one vertex and one loop on it.

Definition 1.7 A *(directed) cycle* in a (di)graph G is a subgraph of G isomorphic to a (directed) cycle graph C_n ($\overrightarrow{C_n}$, respectively).
A *(directed) path* in a (di)graph G is a subgraph of G isomorphic to a (directed) path graph P_n ($\overrightarrow{P_n}$, respectively).
The *length* of a (directed) cycle or path is the number of edges (or arcs) in it.
The two vertices in a path that have degree 1 in the path subgraph are called the *ends* of this path.
A path having ends u and v is called a u-v path.
A *Hamiltonian cycle* of G is a cycle in G with $|V(G)|$ vertices. A graph is called Hamiltonian if it contains a Hamiltonian cycle.
A *Hamiltonian path* of G is a path in G with $|V(G)|$ vertices.

Sometimes the (directed) cycles and paths are identified with their edge (or arc) sets. In Section 3.2, E_1 is considered as $\overrightarrow{P_1}$.

Definition 1.8 A graph is *connected* if a u-v path exists for every two vertices $u \neq v$.
A maximal connected subgraph of a graph G is called a *component* of G. The number of components of G is denoted by $k(G)$.
A *covered component* is a component that is not an isolated vertex. The number of covered components of G is denoted by $c(G)$.
A maximal subgraph of a graph G in which every two edges belong to a cycle is called

a *block* of G.

Definition 1.9 If D is a digraph, the *underlying graph* $G(D)$ is the graph obtained from D by replacing each arc by an edge with the same vertices.

The terms “connected", “component" and “covered component" can be extended to the digraphs by considering their underlying graphs.

Definition 1.10 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs where $V_1 \cap V_2 = \emptyset$. The *(disjoint) union* $G_1 \cup G_2$ of two graphs G_1 and G_2 is $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* $G_1 \vee G_2$ of two graphs G_1 and G_2 is $G_1 \vee G_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup \{\{v,w\} \,|\, v \in V_1, w \in V_2\})$.

Definition 1.11 Let $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ be two digraphs where $V_1 \cap V_2 = \emptyset$. The *(disjoint) union* $D_1 \cup D_2$ of two digraphs D_1 and D_2 is $D_1 \cup D_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The *join* $G_1 \vee G_2$ of two digraphs D_1 and D_2 is $D_1 \vee D_2 = (V_1 \cup V_2, E_1 \cup E_2 \cup V_1 \times V_2)$.

Definition 1.12 A *complete bipartite graph* K_{mn} is the graph $K_{mn} := E_m \vee E_n$.
A *star* S_n is the complete bipartite graph K_{1n}.
A *tree* is a connected graph without cycles.

1.3 Explicit Formulae via Adjacency Matrices

Several explicit formulae for the number of cycles and paths using the adjacency matrices are known. Let $G = (V, E)$ be a simple graph, $V = \{1, \ldots, n\}$. Let $A = A(G) = (a_{ij})_{n,n}$ be the adjacency matrix of G, that is, $a_{ij} = 1$ if vertices i and j are adjacent, otherwise $a_{ij} = 0$. Let $c_k(G)$, $p_k(G)$ denote the number of cycles and paths of length k in G, respectively.
A classical result for counting cycles of given length is given by Harary and Manvel [23].

$$c_3(G) = \frac{1}{6}\mathrm{Tr}(A^3),$$

$$c_4(G) = \frac{1}{8}\left[\mathrm{Tr}(A^4 - 2|E| - 2\sum_{i \neq j} a_{ij}^{(2)})\right],$$

$$c_5(G) = \frac{1}{10}\left[\mathrm{Tr}(A^5) - 5\mathrm{Tr}(A^3) - 5\sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{ij} - 2\right) a_{ii}^{(3)}\right].$$

Closed formulae for $c_6(G)$, $c_7(G)$ and $p_k(G)$, $k = 2,3,4,5,6$ are given in [28] and its referenced papers.
Bax [8] gave an explicit formula for the number of Hamiltonian cycles $c_n(G)$ in G:

$$c_n = \frac{1}{2n}\sum_{S \subseteq V} (-1)^{n-|S|}\mathrm{Tr}(A_S^n),$$

where A_S is the submatrix of A obtained by striking out its rows and columns with the ordinal numbers from the set $S \subseteq V$, and $\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$ is the trace of a matrix A.
The following result due to Perepechko and Voropaev [31] generalizes the formula of Bax and is an explicit expression for $c_k(G)$:

$$c_k(G) = \frac{1}{2k} \sum_{i=2}^{k} (-1)^{k-i} \binom{n-i}{n-k} \sum_{|S|=n-i} \mathrm{Tr}(A_S^k).$$

In a series of recent publications [18–20], Giscard *et al.* proposed a more general theory for counting cycles and paths by constructing a number theory on walks. It also works for multigraphs and digraphs. One of the explicit formulae is

$$c_k(G) = \frac{(-1)^k}{k} \sum_{\substack{S \subseteq V \\ |S| \leq k \\ G[S] \text{ connected}}} \binom{|N_G(S)|}{k-|S|} (-1)^{|S|} \mathrm{Tr}\left(A(G[S])^k\right),$$

where $N_G(S) := \bigcup_{v \in S} N(v) \backslash S$ is the open neighborhood of S in G. For some graphs the computation of $c_k(G)$ can be more efficient by using this formula, since this summation ranges only over small connected induced subgraphs, and the adjacency matrices are small for small k.

2 Cycle and Path Polynomials for Graphs

In this paper, the following edge and vertex local operations for undirected graphs will be used:

- *Edge deletion*. The graph obtained from graph G by removing the edge e is denoted by G_{-e}.
- *Edge contraction*. The graph obtained from G by removing e and unifying the end vertices of e is denoted by $G_{/e}$.
- *Vertex deletion*. The graph obtained from G by removing the vertex v and all its incident edges is denoted by G_{-v}.
- *Edge extraction*. The graph obtained from G by deleting the two end vertices of e is denoted by $G_{\dagger e}$.
- *Edge addition*. The graph obtained from G by adding the edge $\{u,v\}$ is denoted by $G_{+\{u,v\}}$.

2.1 The Cycle Polynomial

Let $G=(V,E)$ be an undirected graph where multiple edges and loops are allowed. We denote the number of cycles of length k in G by $c_k(G)$. A vertex with one loop and two vertices with two parallel edges connecting them are considered as cycle graph C_1 and C_2, respectively. The *cycle polynomial* $\sigma(G)=\sigma(G;x)$ of a graph G is the ordinary generating function of $c_k(G)$, that is

$$\sigma(G)=\sigma(G;x)=\sum_{k=1}^{|V|} c_k(G)x^k.$$

Let $\mathrm{ham}(G)$ denote the number of Hamiltonian cycles in G. Then

$$\sigma(G;x)=\sum_{W\subseteq V} \mathrm{ham}(G[W])x^{|W|}.$$

The cycle polynomial contains information about cycles in a graph. The girth (the length of the shortest cycle) of a graph G is $\min\{i\in\mathbb{N}\,|\,[x^i]\sigma(G;x)>0\}$, and the circumference (the length of the longest cycle) of a graph G is $\deg(\sigma(G;x))$. Graph G is Hamiltonian iff $\deg(\sigma(G;x))=|V(G)|$.
In a graph $G=(V,E)$, an edge $e\in E$ is called a *bridge* if $k(G_{-e})=k(G)+1$. A vertex $v\in V$ is called an *articulation* if $k(G_{-v})>k(G)$.
Evidently, no cycles can lie in more than one component, or more than one block. And each bridge does not belong to any cycle. Then we have

Lemma 2.1 *The cycle polynomial is additive under components and blocks, that is, if two graphs G_1 and G_2 are vertex disjoint or share one vertex, then*

$$\sigma(G_1 \cup G_2; x) = \sigma(G_1; x) + \sigma(G_2; x).$$

Let $G = (V,E)$ be a graph and $e \in E$ a bridge in G, then

$$\sigma(G_{-e}) = \sigma(G).$$

Theorem 2.2 (Edge decomposition) *Let $G = (V,E)$ be a graph. For each edge $e = \{u,v\} \in E$:*

$$\sigma(G) = \begin{cases} x + \sigma(G_{-e}) & \textit{if } e \textit{ is a loop;} \\ \sigma(G_{-e}) + x\left[\sigma(G_{/e}) - \sigma(G_{-u}) - \sigma(G_{-v}) + \sigma(G_{\dagger e})\right] & \textit{otherwise.} \end{cases}$$

Proof: If e is a loop then it is a cycle of length 1 and it belongs to no other cycles. The cycle e of length 1 is counted by x and other cycles are counted by $\sigma(G_{-e})$.
If $e = \{u,v\}$ is not a loop, then G_{-e} contains all the cycles of G without edge e. G_{-u}, G_{-v}, $G_{\dagger e}$ contains the cycles of G without the vertex u, v, and vertices u and v, respectively. Now consider the graph $G_{/e}$. There are three kinds of cycles in G: cycles containing e, cycles not containing e but containing the end vertices u and v of e, and the other cycles. The cycles of the first kind are contained in $G_{/e}$ but their lengths decrease by 1. These cycles are enumerated by $\sigma(G) - \sigma(G_{-e})$. Cycles of the second kind are not contained in $G_{/e}$. Cycles of the third kind are contained in $G_{/e}$. They are exactly the cycles containing at most one end vertex of e, thus they can be enumerated by $\sigma(G_{-u}) + \sigma(G_{-v}) - \sigma(G_{\dagger e})$. Therefore, if e is not a loop, then

$$\sigma(G_{/e}) = \frac{1}{x}\Big(\sigma(G) - \sigma(G_{-e})\Big) + \sigma(G_{-u}) + \sigma(G_{-v}) - \sigma(G_{\dagger e}).$$

□

Lemma 2.3 (Series reduction) *Let $G_1 = (V,E)$ be a graph and $u,v \in V(G_1)$, $G = (V(G_1) \cup \{w\}, E(G_1) \cup \{\{u,w\},\{w,v\}\})$, then*

$$\sigma(G) = (1-x)\sigma(G_1) + x\sigma(G_{1+\{u,v\}}).$$

Proof: $\sigma(G_1)$ counts exactly the cycles of G without the vertex w. And $\sigma(G_{1+\{u,v\}}) - \sigma(G_1)$ counts exactly the cycles of G with the vertex w, where each cycle is counted with one edge less. □

Lemma 2.4 (Parallel reduction) *Let $G = (V,E)$ be a graph, $e = \{u,v\} \in E(G)$ an edge with multiplicity k. We denote by $G_{-(k-1)e}$ the graph obtained from G by removing $k-1$ edges parallel to e, and by G_{-ke} the graph obtained from G by removing all of the k*

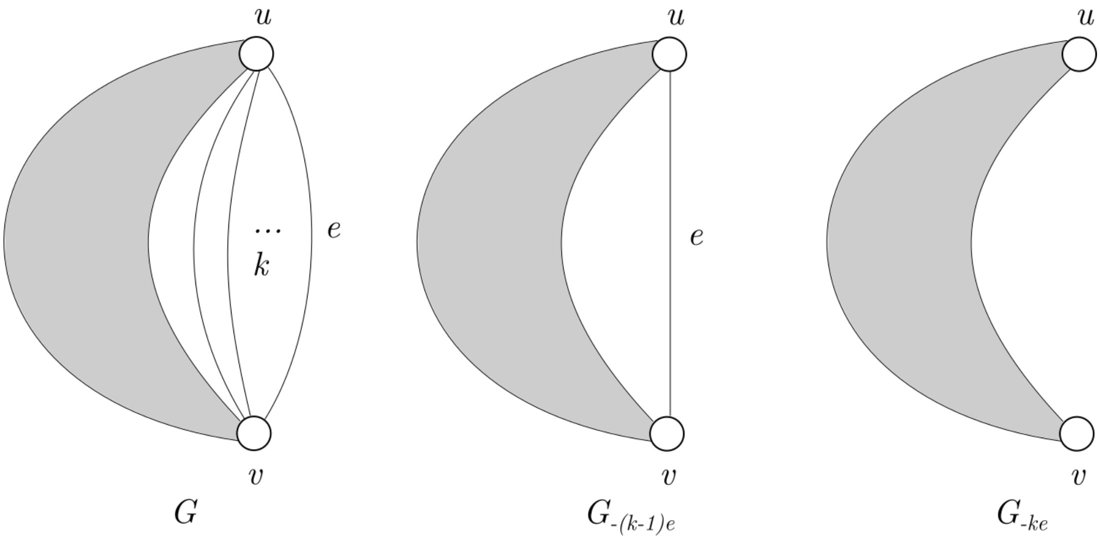


Figure 2.1: Parallel reduction

parallel edges between u and v. Then

$$\sigma(G) = k\sigma(G_{-(k-1)e}) - (k-1)\sigma(G_{-ke}) + \binom{k}{2}x^2.$$

Proof: The number of cycles in G containing one of the edges between u and v equals k times the number of cycles in G containing e but not any edge parallel to e, since e can be replaced by any edge parallel to e and form a different cycle. Therefore, these cycles can be counted by $k[\sigma(G_{-(k-1)e}) - \sigma(G_{-ke})]$. Cycles not containing any edge between u and v can be counted by $\sigma(G_{-ke})$. Cycles containing two edges between u and v are counted by $\binom{k}{2}x^2$. We add these three terms to obtain the reduction formula. □

Theorem 2.5 (Vertex decomposition) *Let $G = (V,E)$ be a graph and $v \in V$ a vertex of G whose incident edges are not multiple. Then*

$$\sigma(G) = \left(1 - \binom{\deg(v)}{2}x\right)\sigma(G_{-v}) + x \sum_{\{u,w\} \in \binom{N(v)}{2}} \sigma(G_{-v+\{u,w\}}).$$

Proof: There are two sorts of cycles in G: the cycles not containing v and the cycles containing v. In the first case they are counted by $\sigma(G_{-v})$. For each cycle of the second case there are exactly two distinct vertices u,w in $N(v)$ such that the edges $\{u,v\}$ and $\{w,v\}$ lie in this cycle. Such cycles can be counted by $x\left[\sigma(G_{-v+\{u,w\}}) - \sigma(G_{-v})\right]$. Thus

$$\begin{aligned}\sigma(G) &= \sigma(G_{-v}) + \sum_{\{u,w\} \in \binom{N(v)}{2}} x\left[\sigma(G_{-v+\{u,w\}}) - \sigma(G_{-v})\right] \\ &= \left(1 - \binom{\deg(v)}{2}x\right)\sigma(G_{-v}) + x \sum_{\{u,w\} \in \binom{N(v)}{2}} \sigma(G_{-v+\{u,w\}}).\end{aligned}$$

□

The definition of the cycle polynomial can also be extended to matroids. It can be defined as the ordinary generating function for the circuits of a matroid. Evidently it is a matroid invariant and the cycle polynomial of a graph G is the cycle polynomial of the cycle matroid $M(G)$ of G. A *bond* $F \subseteq E(G)$ of a graph G is a minimal edge subset such that G_{-F} has more components than G. Since the dual of the cycle matroid of a graph G is the bond matroid of G, we have the following theorem.

Theorem 2.6 *Let G be a planar graph and $b_k(G)$ be the number of bonds of k edges in G, and $B(G;x) = \sum_k b_k(G)x^k$ be the ordinary generating function for $b_k(G)$. Then*

$$B(G;x) = \sigma(G^\star;x),$$

where $G^\star$ is the dual graph of G.

Let G and H be any two simple graphs, recall that a *homomorphism* of G to H is a mapping $f : V(G) \to V(H)$ such that $\{f(u), f(v)\} \in E(H)$ if $\{u,v\} \in E(G)$. The number of homomorphisms of G to H is denoted by $\hom(G,H)$. If G and H are multigraphs with loops, a homomorphism of G to H is a function $f_V : V(G) \to V(H)$ together with an associated function $f_E : E(G) \to E(H)$ consistent with f_V in that $f_E(\{u,v\}) = \{f_V(u), f_V(v)\}$. Then we have the following expression of $\hom(G,H)$:

$$\hom(G,H) = \sum_{f:V(G)\to V(H)} \prod_{u,v\in V(G)} m(\{f(u), f(v)\})^{m(\{u,v\})},$$

where $m(e)$, $e \in E(G)$ denotes the multiplicity of e in G. Let $\mathrm{inj}(G,H)$ denote the number of injective homomorphisms of G to H. For a partition $\pi \in \Pi(V(G))$ of the vertex set of G, let G/π denote the graph obtained from G by identifying the vertices that belong to the same block of π. Then

$$\hom(G,H) = \sum_{\pi\in\Pi(V(G))} \mathrm{inj}(G/\pi, H).$$

The following formula can be obtained by applying the Möbius inversion of partition lattice

$$\mathrm{inj}(G,H) = \sum_{\pi\in\Pi(V(G))} \mu(\widehat{0},\pi)\hom(G/\pi, H),$$

where

$$\mu(\widehat{0},\pi) = (-1)^{|V(G)|-|\pi|} \prod_{B\in\pi} (|B|-1)!$$

is the Möbius function of the partition lattice $\Pi(V(G))$ and $\widehat{0} = \{\{v\} \,|\, v \in V(G)\}$.
Let $\mathrm{aut}(G)$ denote the number of *automorphisms* of G, that is, the number of isomorphisms from G to G. And let $\binom{G}{H}$ denote the number of isomorphic copies of H contained in G, then we have

$$\binom{G}{H} = \frac{\mathrm{inj}(H,G)}{\mathrm{aut}(H)}.$$

Then we have the following expression for the cycle polynomial

$$\begin{aligned}\sigma(G;x) &= \sum_{n\geq 1}\binom{G}{C_n}x^n \\ &= \sum_{n=1}^{|V(G)|}\frac{\mathrm{inj}(C_n,G)}{\mathrm{aut}(C_n)}x^n \\ &= \sum_{n=1}^{|V(G)|}\frac{\mathrm{inj}(C_n,G)}{2n}x^n,\end{aligned}$$

because the automorphism group of the cycle graph C_n is the dihedral group D_n, whose order is $2n$. A variety of results of graph homomorphisms and more efficient ways to compute $\mathrm{inj}(G,H)$ can be found in [3, 17, 24].

Further results for the cycle polynomial can be found in Section 2.3.

2.2 The Path Polynomial

Let $G = (V,E)$ be an undirected graph. Let $p_k(G)$ denote the number of paths of length k in G. Similarly to the cycle polynomial, the *path polynomial* $\pi(G) = \pi(G;x)$ of the graph G can be defined as the ordinary generating function of $p_k(G)$, i.e.

$$\pi(G) = \pi(G;x) = \sum_{k=1}^{|V|-1} p_k(G)x^k.$$

For $u,v \in V(G)$ we denote the number of paths of length k in G with one of the two end vertices u and with two end vertices u,v by $p_k(G,u)$ and $p_k(G,u,v)$, respectively. We define

$$\pi_u(G;x) = \sum_{k=1}^{|V|-1} p_k(G,u)x^k,$$

and

$$\pi_{uv}(G;x) = \sum_{k=1}^{|V|-1} p_k(G,u,v)x^k.$$

Then we have

$$\pi(G;x) = \frac{1}{2}\sum_{v\in V} \pi_v(G;x),$$

and

$$\pi(G;x) = \sum_{\{u,v\}\in\binom{V}{2}} \pi_{uv}(G;x),$$

and

$$\pi_v(G;x) = \sum_{u\in V} \pi_{vu}(G;x).$$

Because $\sigma(G_{+\{u,v\}}) - \sigma(G)$ counts the cycles of $G_{+\{u,v\}}$ containing edge $\{u,v\}$, which are exactly u-v paths of G together with edge $\{u,v\}$, we have

$$\pi_{uv}(G;x) = \frac{1}{x}\left(\sigma(G_{+\{u,v\}};x) - \sigma(G;x)\right).$$

Let $\mathrm{hampath}(G)$ denote the number of Hamiltonian paths of G. We have

$$\pi(G;x) = \sum_{W\subseteq V} \mathrm{hampath}(G[W])x^{|W|-1}.$$

The preceding notations can be applied to simplify the computation of $\pi(G)$ and $\sigma(G)$ in some cases.

Theorem 2.7 *Let G be a graph and u,v a separating vertex pair of G, that is, $G_1 \cup G_2 = G$, $V_1 \cap V_2 = \{u,v\}$, and $E_1 \cap E_2 = \emptyset$. Then*

$$\sigma(G) = \sigma(G_1) + \sigma(G_2) + \pi_{uv}(G_1)\pi_{uv}(G_2).$$

Proof: Cycles lying in G_1 are counted by $\sigma(G_1)$, cycles lying in G_2 are counted by $\sigma(G_2)$. Other cycles of G are unions of any u-v path in G_1 and any u-v path in G_2, they are counted by $\pi_{uv}(G_1)\pi_{uv}(G_2)$. □

Theorem 2.8 *Let G be a graph and u an articulation of G, that is, $G_1 \cup G_2 = G$, $V_1 \cap V_2 = \{u\}$ and $E_1 \cap E_2 = \emptyset$. Then*

$$\pi(G) = \pi(G_1) + \pi(G_2) + \pi_u(G_1)\pi_u(G_2).$$

Proof: $\pi(G_1)$ and $\pi(G_2)$ count paths of G lying only in G_1 and G_2, respectively. Each of other paths of G is the union of a path in G_1 with end u and a path in G_2 with end u. They are counted by $\pi_u(G_1)\pi_u(G_2)$. □

$\pi_u(G)$ can be computed recursively:

Theorem 2.9

$$\pi_u(G) = x \sum_{v \in N(u)} \pi_v(G_{-u}) + \deg(u) \cdot x.$$

Proof: $\pi_u(G)$ is the generating function for the number of paths of G with an end u. There are exactly $\deg(u)$ such paths of length 1. Each such path of length greater than 1 contains exactly one edge $\{u,v\}$ incident to u, and the remaining part of this path can be any path in G_{-u} with the end v. □

We also have the parallel reduction for the path polynomial similar to Lemma 2.4. The proof is the same as the proof of Lemma 2.4 except that the term $\binom{k}{2}x^2$ is not needed.

Lemma 2.10 (Parallel reduction) *Let $G = (V,E)$ be a graph, $e = \{u,v\} \in E(G)$ an edge with multiplicity k. We denote by $G_{-(k-1)e}$ the graph obtained from G by removing $k-1$ edges parallel to e, and by G_{-ke} the graph obtained from G by removing all of the k parallel edges between u and v. Then*

$$\pi(G) = k\pi(G_{-(k-1)e}) - (k-1)\pi(G_{-ke}).$$

Theorem 2.11 (Edge decomposition) *Let $G = (V,E)$ be a graph and $e = \{u,v\} \in E$ an edge of G. If e is a loop then $\pi(G) = \pi(G_{-e})$. If e is not a loop, then*

$$\pi(G) = \pi(G_{-e}) - x\pi(G_{-u}) - x\pi(G_{-v}) + x\pi(G_{\dagger e}) \\ + x\pi(G_{/e}) + x\pi_u(G_{-v}) + x\pi_v(G_{-u}) + x.$$

Proof: The following table shows, which paths in G are counted by path polynomials of which graphs.

Paths	G_{-e}	G_{-v}	G_{-u}	$G_{\dagger e}$	$G_{/e}$	other
containing only e						x
containing e, and u,v are not ends					$\checkmark^*$	
containing e, and u is end, v not						$x\pi_u(G_{-v})$
containing e, and v is end, u not						$x\pi_v(G_{-u})$
containing neither e nor u nor v	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	$\checkmark$	
containing v, but neither e nor u	$\checkmark$		$\checkmark$		$\checkmark$	
containing u, but neither e nor v	$\checkmark$	$\checkmark$			$\checkmark$	
containing u and v, but not containing e	$\checkmark$					

The cell marked with * means, paths of G containing e whose ends are neither v nor u are counted by $G_{/e}$ but their lengths are counted one less. The edge decomposition formula follows by summarizing all the cases. □

Theorem 2.12 (Vertex decomposition) *Let $G=(V,E)$ be a graph. For each $v \in V$, if $\deg(v) \geqslant 2$, then*

$$\pi(G) = x \sum_{\{u,w\}\in\binom{N(v)}{2}} \pi(G_{-v+\{u,w\}}) + \left(1 - \binom{\deg(v)}{2} x\right) \pi(G_{-v}) + \pi_v(G).$$

If $\deg(v)=1$, $\{u,v\}\in E$, then

$$\pi(G) = \pi(G_{-v}) + x\pi_u(G_{-v}) + x.$$

Proof: If $\deg(v)=1$, each path in G contains either v or not. Paths containing v are counted by $\pi_v(G) = x\pi_u(G_{-v}) + x$, which is a special case of Proposition 2.9. Paths not containing v are counted by $\pi(G_{-v})$.
If $\deg(v) \geq 2$, each path in G belongs to exactly one of the following three classes:

1. paths not containing v,

2. paths containing v as an end,

3. paths containing v, and v is not their end.

The first two classes are counted by $\pi(G_{-v})$ and $\pi_v(G)$, respectively. Because of the same argument as in the proof of Theorem 2.5, the class 3 are counted by

$$x \sum_{\{u,w\}\in\binom{N(v)}{2}} \left[\pi(G_{-v+\{u,w\}}) - \pi_v(G)\right].$$

We conclude the theorem by adding these terms together. □

For $1 \leq k \leq n-1$, the path polynomials of basic graph classes are derived as follows. Obviously there are $n-k$ paths of length k in P_n, and n paths of length k in C_n. Paths of length k in K_n correspond to ordered selections of k out of n vertices without repetition, however, each ordered selection and its inverse correspond to the same path, thus there are $\frac{1}{2}n^{\underline{k+1}}$ paths. In S_n there are clearly $\binom{n}{2}$ paths of length 2 and n paths of length 1. We conclude the following theorem.

Theorem 2.13

$$\pi(P_n) = \sum_{k=1}^{n-1}(n-k)x^k,$$

$$\pi(C_n) = n\sum_{k=1}^{n-1}x^k,$$

$$\pi(S_n) = \binom{n}{2}x^2 + nx,$$

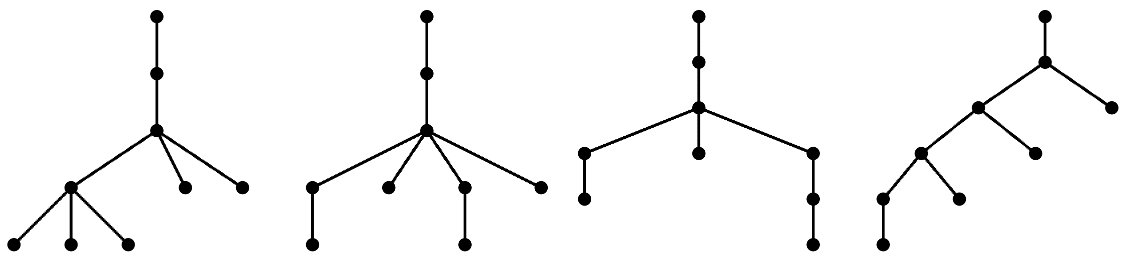
$$\pi(K_n) = \frac{1}{2}\sum_{k=1}^{n-1}n^{\underline{k+1}}x^k,$$

where P_n, C_n, S_n and K_n are path graphs, cycle graphs, star graphs and complete graphs, respectively.

The distinguishing power of the path polynomial is distinct from other polynomials. K_3 and $K_{1,3}$ are the smallest pair non-isomorphic graphs with the same path polynomial

$$\pi(K_3;x) = \pi(K_{1,3};x) = 3x^2 + 3x.$$

The path polynomial can distinguish all non-isomorphic trees with up to 8 vertices. There are two pairs of non-isomorphic trees on 9 vertices with the same path polynomials, which is shown by following figures:



T_1 T_2 T_3 T_4

$$\pi(T_1;x) = \pi(T_2;x) = 3x^4 + 12x^3 + 13x^2 + 8x,$$
$$\pi(T_3;x) = \pi(T_4;x) = 2x^5 + 6x^4 + 10x^3 + 10x^2 + 8x.$$

In [21], an elementary procedure for constructing n pairwise non-isomorphic caterpillars (trees in which all of the non-leaf vertices form a path) with the same path polynomial are given.

The path polynomial can also be expressed by graph homomorphisms. Because $\mathrm{aut}(P_n) = 2$ for all path graphs $P_n, n \geq 2$, we have

$$\pi(G;x) = \sum_{n=1}^{|V(G)|-1}\binom{G}{P_{n+1}}x^n = \frac{1}{2}\sum_{n=1}^{|V(G)|-1}\mathrm{inj}(P_{n+1},G)x^n.$$

Let $G = (V,E)$ be a simple graph. The *line graph* $L(G)$ of G is a graph such that $V(L(G)) = E$ and two vertices of $L(G)$ are adjacent iff they are incident in G. In [36], Whitney proved that with the exception of C_3 and $K_{1,3}$, any two connected simple graphs with isomorphic line graphs are isomorphic. It is well known that $L(K_{1,3}) \cong C_3$, $L(C_n) \cong C_n$ for $n \geq 3$ and $L(P_n) \cong P_{n-1}$ for $n \geq 2$. Observe that for $F \subseteq E$:

- F forms a path of length k iff $L(G)[F] \cong P_k$, $k \geq 1$;
- F forms a cycle of length k iff $L(G)[F] \cong C_k$, $k \geq 4$;
- F forms a cycle of length 3 or a claw (that is, $K_{1,3}$) iff $L(G)[F] \cong C_3$.

Because of this observation, the ordinary generating function for the number of induced cycles and paths in a line graph can be obtained from the cycle polynomial and the path polynomial, respectively. The number of edge subsets F of G such that $(\bigcup_{e\in F} e, F) \cong K_{1,3}$ is $\sum_{v\in V} \binom{\deg(v)}{3}$, because each F consists of three edges incident to one vertex. We obtain the following result.

Theorem 2.14 *Let G be a simple graph. Then*

$$\sum_{\substack{S\subseteq V(L(G))\\ \exists k\geq 3:\, L(G)[S]\cong C_k}} x^{|S|} = \sigma(G;x) + x^3 \cdot \sum_{v\in V} \binom{\deg(v)}{3}$$

and

$$\sum_{\substack{S\subseteq V(L(G))\\ \exists k\geq 1:\, L(G)[S]\cong P_k}} x^{|S|} = \pi(G;x).$$

The reason for $\pi(K_3;x) = \pi(K_{1,3};x)$ is $L(K_3) \cong L(K_{1,3}) \cong K_3$.

2.3 The Bivariate Cycle Polynomial

The cycle polynomial can be generalized. We count now the disjoint union of cycles instead of one cycle. We introduce another variable to count the number of cycle components. The *bivariate cycle polynomial* is defined as

$$\widehat{\sigma}(G) = \widehat{\sigma}(G;x,y) = \sum_{\substack{F\subseteq E \\ \forall v\in V:\deg_{G\langle F\rangle}(v)=2 \text{ or } 0}} x^{|F|}y^{c(G\langle F\rangle)},$$

where $c(G)$ denotes the number of covered components, i.e. components that are not isolated vertices, of G.
One advantage of this generalization is that this polynomial is multiplicative under components and blocks rather than additive. The bivariate cycle polynomial of an arbitrary tree, or a null graph $(\emptyset,\emptyset)$, equals the multiplicative identity 1.
The cycle polynomial can be obtained from the bivariate cycle polynomial by

$$\sigma(G,x) = [y^1]\widehat{\sigma}(G;x,y).$$

The number of vertex disjoint cycle covers $cc(G,k)$ of G with k cycles is

$$cc(G,k) = [x^{|V|}y^k]\widehat{\sigma}(G;x,y).$$

If G is a multigraph with loops, we can calculate $\widehat{\sigma}(G)$ through the bivariate cycle polynomials of some simple graphs according to the following theorems.

Theorem 2.15 *Let G be a graph and $e=\{v,v\}$ a loop of G with multiplicity k. The graph obtained from G by deleting all k multiple loops on v is denoted by G_{-ke}. Then*

$$\widehat{\sigma}(G;x,y) = \widehat{\sigma}(G_{-ke};x,y) + kxy\widehat{\sigma}(G_{\dagger e};x,y).$$

Theorem 2.16 *Let G be a graph and $e=\{u,v\}, u\neq v$ an edge of G with multiplicity k. Then*

$$\widehat{\sigma}(G) = k\widehat{\sigma}(G_{-(k-1)e}) - (k-1)\widehat{\sigma}(G_{-ke}) + \binom{k}{2}x^2y\widehat{\sigma}(G_{\dagger e}).$$

The proofs of these two theorems are similar to the proofs of the corresponding theorems of the univariate version.
If G is loopless, then $\widehat{\sigma}(G)$ can be rewritten as a summation over partitions of the vertex set

$$\widehat{\sigma}(G) = \sum_{\pi\in\Pi(V)} \prod_{\substack{W\in\pi \\ |W|\neq 1}} \operatorname{ham}(G[W])x^{|W|}y.$$

This formula can be applied to compute the bivariate cycle polynomial of complete graphs and complete bipartite graphs. The number of Hamiltonian cycles of the complete graph K_n on n vertices is $\frac{1}{2}(n-1)!$. Then we have

$$\widehat{\sigma}(K_n) = \sum_{\pi \in \Pi_n} \prod_{\substack{W \in \pi \\ |W| \neq 1}} \left\lfloor \frac{1}{2}(|W|-1)! \right\rfloor x^{|W|} y.$$

Hamiltonian cycles are the building bricks of graphs whose covered components are cycles. The reason for the floor function and the condition $|W| \neq 1$ is as follows. The blocks of size 2 are bricks of no such graphs, and the blocks of size 1 play the role of isolated vertices in graphs, thus they contribute the terms 0 and 1 in each product, respectively. Applying now the exponential formula, we get

$$\begin{aligned}\sum_{n\geq 1} \widehat{\sigma}(K_n;x,y)\frac{z^n}{n!} &= \exp\left(z + \sum_{n\geq 3} \frac{1}{2n} x^n y z^n\right) \\ &= \exp\left(z - \frac{y}{2}\log(1-xz) - \frac{xyz}{2} - \frac{x^2yz^2}{4}\right).\end{aligned}$$

From the previous formula $\widehat{\sigma}(K_n)$ can be computed in polynomial time in n.
Now we calculate $\widehat{\sigma}(K_{mn}) = \widehat{\sigma}(K_{mn};x,y)$ for the complete bipartite graph $K_{mn} = (M \cup N, \big\{\{w,v\} \big| w \in M \wedge v \in N\big\})$, where $|M| = m$, $|N| = n$, and $M \cap N = \emptyset$. Without loss of generality, we assume $m \leq n$. It is not difficult to see,

$$\sigma(K_{mn};x) = \sum_{k=2}^{m} \binom{m}{k}\binom{n}{k}\frac{1}{2}k!(k-1)!x^{2k}.$$

Now we partition the vertex set. Because K_{mn} has only even cycles and each cycle has the vertices of the same numbers from M and N, we can take the summation over the partitions of set M. For each summand we multiply the numbers of Hamiltonian cycles of complete bipartite subgraphs, where each one is induced by a block of the partition and a subset of N of the same size. Furthermore it must be multiplied with a multinomial coefficient, that is, the number of ways to split the set N into the blocks that form the parts of cycles and a block of isolated vertices.

$$\begin{aligned}\widehat{\sigma}(K_{mn}) &= \sum_{\pi \in \Pi_m} \frac{n!}{(n-m+|\{W \in \pi \big| |W| = 1\}|)! \prod_{\substack{W\in\pi \\ |W|\neq 1}} |W|!} \times \\ &\qquad\qquad \times \prod_{\substack{W \in \pi \\ |W| \neq 1}} \frac{1}{2}|W|!(|W|-1)!x^{2|W|}y \\ &= \sum_{\pi \in \Pi_m} \frac{n!}{(n-m+|\{W \in \pi \big| |W| = 1\}|)!} \prod_{\substack{W \in \pi \\ |W| \neq 1}} \frac{1}{2}(|W|-1)!x^{2|W|}y.\end{aligned}$$

Since this summation depends only on the type of partition $\pi \in \Pi_m$, $\widehat{\sigma}(K_{mn})$ can be represented as a sum over number partitions λ of m, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s) =$

$(1^{k_1}\dots m^{k_m})$, in order to reduce the terms of summation.

$$\widehat{\sigma}(K_{mn}) = \sum_{\lambda\vdash m} \frac{1}{k_1!\cdots k_m!}\binom{m}{\lambda}\frac{n!}{(n-m+k_1)!}\prod_{i=2}^{m}\left(\frac{1}{2}(i-1)!x^{2i}y\right)^{k_i}.$$

The bivariate cycle polynomial $\widehat{\sigma}(G;x,y)$ satisfies the same edge and vertex decomposition formulae as the univariate version. Their proofs are similar.

Theorem 2.17 *Let $G=(V,E)$ be a graph. For each edge $e=\{u,v\}\in E$:*

$$\widehat{\sigma}(G) = \begin{cases} xy\widehat{\sigma}(G_{\dagger e})+\widehat{\sigma}(G_{-e}) & \textit{if } e \textit{ is a loop;}\\ \widehat{\sigma}(G_{-e})+x\left[\widehat{\sigma}(G_{/e})-\widehat{\sigma}(G_{-u})-\widehat{\sigma}(G_{-v})+\widehat{\sigma}(G_{\dagger e})\right] & \textit{otherwise.}\end{cases}$$

Theorem 2.18 *Let $G=(V,E)$ be a graph and $v\in V$ a vertex of G whose incident edges are not multiple. Then*

$$\widehat{\sigma}(G) = \left(1-\binom{\deg(v)}{2}x\right)\widehat{\sigma}(G_{-v})+x\sum_{\{u,w\}\in\binom{N(v)}{2}}\widehat{\sigma}(G_{-v+\{u,w\}}).$$

The reconstruction conjecture is a famous unsolved problem in graph theory. The multiset $\{G_{-v}\mid v\in V(G)\}$ is called the *deck* of a graph G. This conjecture proposes that every graph with at least three vertices can be uniquely reconstructed from its deck. If some property of G can be uniquely determined from the deck of G, it is said to be *reconstructible*.

If G and H are graphs, let $\binom{G}{H}$ denote the number of subgraphs of G isomorphic to H. If $|V(H)|<|V(G)|$, then $\binom{G}{H}$ is reconstructible [26]. In the case of $|V(H)|=|V(G)|$, if H is disconnected, then $\binom{G}{H}$ is also reconstructible [27]. The number of Hamiltonian cycles is also shown to be reconstructible [34].

We conclude the following theorem from these facts.

Theorem 2.19 *The bivariate cycle polynomial $\widehat{\sigma}(G;x,y)$ is reconstructible from the graph deck of G.*

3 Cycle and Path Polynomials for Digraphs

In [13], Chung and Graham introduced a bivariate digraph polynomial called the *cover polynomial* which has a Tutte-like deletion-contraction recurrence relation. It is one of the well-researched digraph polynomials. The research on digraph polynomials counting paths and cycles is motivated by the cover polynomial.

3.1 The Cycle Polynomial and the Path Polynomial of Digraphs

In this paper, the following arc operation for (multi-)digraphs will be used:

- *Arc deletion*. The graph obtained from D by removing the arc e is denoted by D_{-e}.
- *Arc contraction*. If $e=(u,v)\in E$, $u\neq v$, $D_{/e}$ is defined as the digraph obtained from D by unifying the two vertices u and v into a new vertex w, and removing exactly the arcs of the form (u,x) or (y,v) from E. If $e=(u,u)$, the vertex u is also removed.
- *Arc extraction*. For $e=(u,v)$, $D_{\dagger e}$ is defined as the digraph obtained from D by removing u and v and their (or its if $u=v$) incident arcs.
- *Arc addition*. The graph obtained from D by adding the arc $(u,v),u,v\in V$ is denoted by $D_{+(u,v)}$.

A digraph D on the vertex set $V(D)=\{1,\dots,n\}$ can be represented as a matrix $A=(a_{ij})\in\mathbb{N}^{n\times n}$, where a_{ij} is the number of arcs from vertex i to vertex j. That is, A is the adjacency matrix of D.
The digraph operations can be expressed as the matrix operations. Let $D=(V,E)$ be a digraph and $A(D)$ be the adjacency matrix of D. Without loss of generality, let $V=\{1,\dots,n\}$. Then for $e=(i,j)\in E$:

- $A(D_{-e})$ can be obtained from $A(D)$ by subtracting 1 from a_{ij},
- $A(D_{/e})$ can be obtained from $A(D)$ by first exchanging row i and row j then deleting row j and column j,
- $A(D_{\dagger e})$ can be obtained from $A(D)$ by deleting row i, row j, column i and column j, and
- $A(D_{+(i,j)})$ can be obtained from $A(D)$ by adding 1 to a_{ij}.

Let $D=(V,E)$ be a digraph where multiple arcs and loops are allowed. The *cycle*

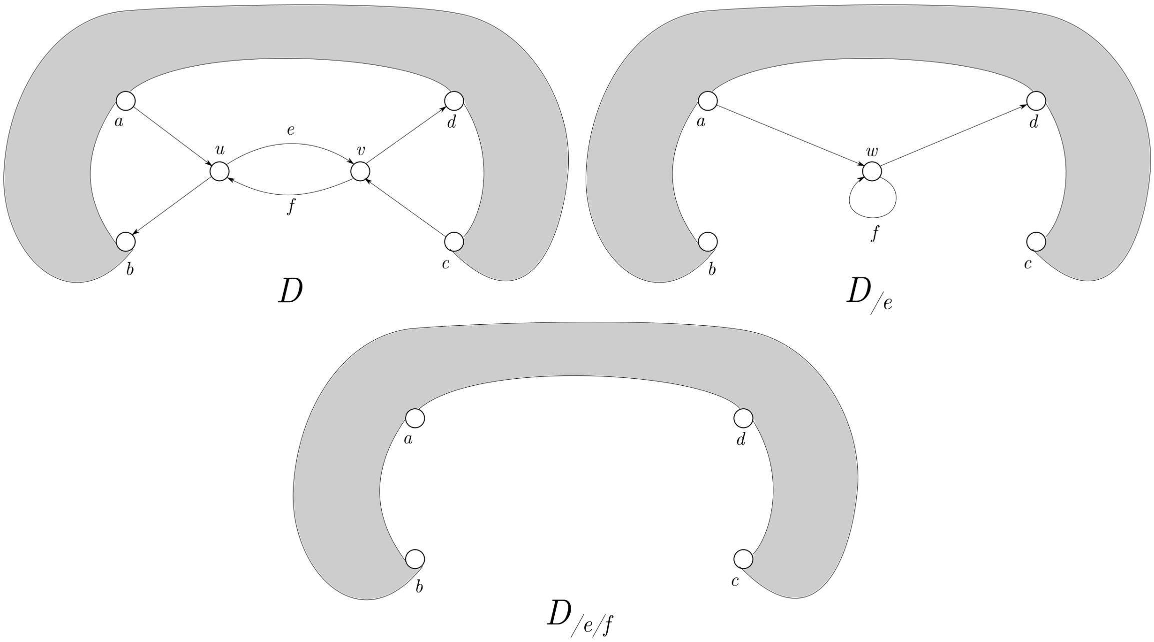


Figure 3.1: Arc contraction on a digraph

polynomial $\sigma(D) = \sigma(D;x)$ of the digraph D is defined as

$$\sigma(D) = \sigma(D;x) = \sum_{k=1}^{|V|} c_k(D)x^k,$$

where $c_k(D)$ denotes the number of directed cycles of length k in D. Similarly, the *path polynomial* of D is defined as

$$\pi(D) = \pi(D;x) = \sum_{k=1}^{|V|-1} p_k(D)x^k,$$

where $p_k(D)$ denotes the number of directed paths of length k in D.
The cycle polynomial and the path polynomial of digraphs satisfy respectively the following recurrence relations:

Theorem 3.1 *If $D = (V,E)$ is a digraph and $e \in E$ is an arc of D, then*

$$\sigma(D) = \begin{cases} \sigma(D_{-e}) + x & \textit{if e is a loop,} \\ \sigma(D_{-e}) + x\sigma(D_{/e}) - x\sigma(D_{\dagger e}) & \textit{if e is not a loop,} \\ & \textit{and there are no loops on u or v.} \end{cases}$$

Proof: If e is a loop, it is counted by x and other cycles are counted by $\sigma(D_{-e})$. If e is not a loop and there are no loops on u or v, $\sigma(D_{-e})$ counts exactly directed cycles in D without e. $D_{/e}$ contains exactly all cycles of D containing e with lengths decreased by 1, and all cycles of $D_{\dagger e}$. Hence $x[\sigma(D_{/e}) - \sigma(D_{\dagger e})]$ counts exactly directed cycles of D

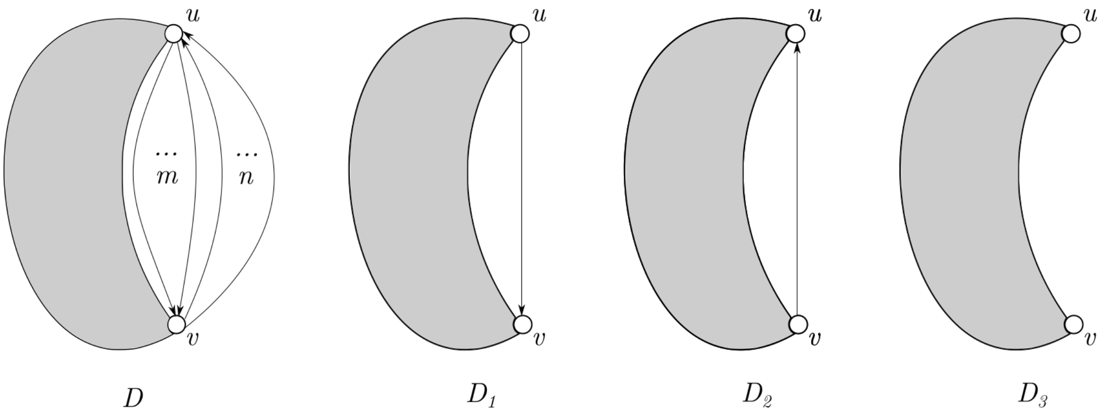


Figure 3.2: Simplification of parallel and anti-parallel arcs

containing e. □

The recurrence for the path polynomial is similar. If e is a loop it does not belong to any directed path and so can be deleted, but if e is not a loop, x must be added in order to count the directed path e. Then we have the following recurrence.

Theorem 3.2 *If $D = (V,E)$ is a digraph and $e \in E$ is an arc of D, then*

$$\pi(D) = \begin{cases} \pi(D_{-e}) & \text{if } e \text{ is a loop,} \\ \pi(D_{-e}) + x\pi(D_{/e}) - x\pi(D_{\dagger e}) + x & \text{if } e \text{ is not a loop.} \end{cases}$$

We can also transform a digraph into several digraphs in order to ensure that there is at most one arc between each pair of vertices. Its proof is similar to Lemma 2.4.

Theorem 3.3 *Let $D = (V,E)$ be a digraph and $u, v \in V$. Suppose that the arc (u,v) has the multiplicity n and the arc (v,u) has the multiplicity m in E. Let D_3 be the digraph obtained from D by deleting all of the n arcs (u,v) and the m arcs (v,u), and let $D_1 = D_{3+(u,v)}, D_2 = D_{3+(v,u)}$, then*

$$\sigma(D) = n\sigma(D_1) + m\sigma(D_2) - (n+m-1)\sigma(D_3) + nmx^2,$$

and

$$\pi(D) = n\pi(D_1) + m\pi(D_2) - (n+m-1)\pi(D_3).$$

Now two vertex operations for digraphs need to be defined in order to state the vertex decomposition formulae. Given is a digraph $D = (V,E)$ and $v \in V$, the sets $N^+(v) := \{u \in V \big| (v,u) \in E\}$ and $N^-(v) := \{u \in V \big| (u,v) \in E\}$ are called the *out-neighborhood* and the *in-neighborhood* of v in D, respectively.

- Vertex deletion. The digraph obtained from D by removing the vertex v and all its incident arcs is denoted by D_{-v}.
- Vertex contraction. If the arcs incident with v are not multiple, $D_{/e}$ is defined

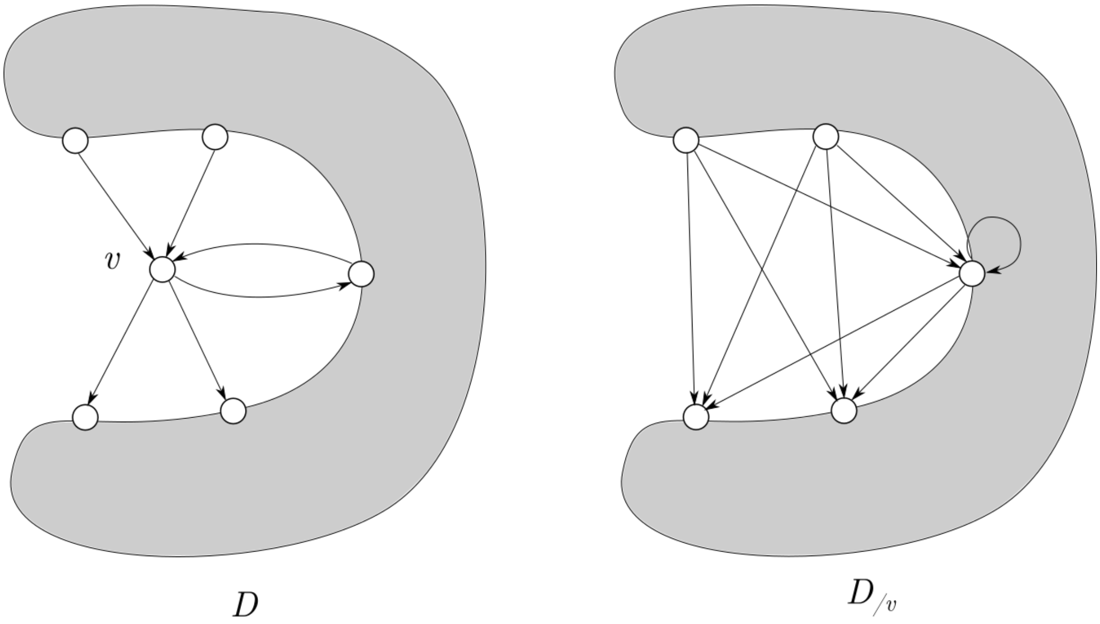


Figure 3.3: Vertex contraction on a digraph

as the digraph obtained from D_{-v} by adding the arcs of $N^{-}(v) \times N^{+}(v)$. For a multidigraph, the multiplicity of an added arc (u,w) equals the multiplicity of (u,v) times the multiplicity of (v,w).

Given is a digraph $D=(V,E)$ and $v \in V$, $E^{-}(v)$ and $E^{+}(v)$ are defined to be the sets of arcs with head v and tail v, respectively, that is, $E^{+}(v) := \{e=(v,u) \in E\}$ and $E^{-}(v) := \{e=(u,v) \in E\}$. We call $\deg^{+}(v) := |E^{+}(v)|$ the *out-degree* and $\deg^{-}(v) := |E^{-}(v)|$ the *in-degree* of v. The number of directed paths of length k beginning with v in D is denoted by $p_k(D,v,+)$. Similarly, the number of directed paths of length k ending with v in D is denoted by $p_k(D,v,-)$. We define the ordinary generating functions for $p_k(D,v,+)$ and $p_k(D,v,-)$:

$$\pi_{v+}(D) := \sum_{k=1}^{|V|-1} p_k(D,v,+)x^k$$

and

$$\pi_{v-}(D) := \sum_{k=1}^{|V|-1} p_k(D,v,-)x^k.$$

$\pi_{v+}(D)$ and $\pi_{v-}(D)$ satisfy the same decomposition formula as Theorem 2.9.

Theorem 3.4 *Let $D=(V,E)$ be a digraph and $v \in V$, then*

$$\pi_{v+}(D) = \deg^{+}(v) \cdot x + x \sum_{u:(v,u) \in E^{+}(v)} \pi_{u+}(D_{-v}),$$

and

$$\pi_{v-}(D) = \deg^{-}(v) \cdot x + x \sum_{u:(u,v) \in E^{-}(v)} \pi_{u-}(D_{-v}).$$

Then we have the vertex decomposition formulae for $\sigma(D)$ and $\pi(D)$.

Theorem 3.5 *Let $D=(V,E)$ be a digraph and $v \in V$, then we have*

$$\sigma(D)=(1-x)\sigma(D_{-v})+x\sigma(D_{/v}),$$

and

$$\pi(D)=(1-x)\pi(D_{-v})+x\pi(D_{/v})+\pi_{v+}(D)+\pi_{v-}(D).$$

Proof: $D_{/v}$ contains exactly cycles and paths of D not containing v, and cycles and paths of D containing v but v is neither source or sink of a path, with length decreased by 1. □

The decomposition formulae for digraphs are easier than that for graphs. There are also relationships between the digraph version and graph version of these polynomials.

Theorem 3.6 *Let $D(G)$ denote the digraph obtained from the undirected graph G by replacing each edge $\{u,v\} \in E$ by two oppositely oriented arcs (u,v) and (v,u). Then we have*

$$\pi(D(G))=2\pi(G),$$

and

$$\sigma(D(G))=2\sigma(G)+|E(G)|x^2.$$

Proof: Each path or cycle of G corresponds to two directed paths or cycles of different directions in $D(G)$. It is easy to see, directed cycles or paths of G arising from different cycles or paths of G are different, and all directed cycles or paths of $D(G)$ arise from corresponding cycles and paths of G except the $|E|$ cycles consisting of two arcs arising from one edge of G. □

3.2 The Cover Polynomials

The *cover polynomial* $C(D;x,y)$ introduced in [13] is defined recursively:

$$C(D;x,y) = \begin{cases} C(D_{-e};x,y) + yC(D_{/e};x,y) & \text{if } e \text{ is a loop,} \\ C(D_{-e};x,y) + C(D_{/e};x,y) & \text{if } e \text{ is not a loop,} \end{cases}$$

and $C(E_n;x,y) = x^{\underline{n}}$ for arc-less digraph E_n, where $x^{\underline{n}} := x(x-1)\dots(x-i+1)$, $x^{\underline{0}} := 1$ is the falling factorial.
The combinatorial interpretation of $C(D;x,y)$ is

$$C(D;x,y) = \sum_{i,j} c_{i,j}(D) x^{\underline{i}} y^j,$$

where $c_{i,j}(D)$ denotes the number of ways of disjointly covering all the vertices of D with i directed paths and j directed cycles. (Notice that isolated vertices are regarded as directed paths of length 0 by the cover polynomial and the following geometric cover polynomial.)
The cover polynomial is a digraph polynomial with Tutte-like deletion-contraction recursion. It is also a generalization of the rook polynomial. For the counting of cycle-path covers of a digraph, the “normal" power can be used instead of the falling factorial. The *geometric cover polynomial* introduced in [14] is the ordinary generating function for $c_{i,j}(D)$

$$\widetilde{C}(D;x,y) = \sum_{i,j} c_{i,j}(D) x^i y^j.$$

It satisfies the same recurrence relation as the cover polynomial, but the initial condition is $\widetilde{C}(E_n;x,y) = x^n$.

For two (disjoint) digraphs D_1 and D_2 there are

$$\widetilde{C}(D_1 \cup D_2;x,y) = \widetilde{C}(D_1;x,y)\widetilde{C}(D_2;x,y)$$

and

$$C(D_1 \vee D_2;x,y) = C(D_1;x,y)C(D_2;x,y),$$

where $D_1 \vee D_2 := (V(D_1) \cup V(D_2), E(D_1) \cup E(D_2) \cup V(D_1) \times V(D_2))$ is the join of two digraphs.

A bijective proof of the multiplicity of the cover polynomial with respect to join will be stated here.

Theorem 3.7 *[13] For any two disjoint digraphs D_1 and D_2,*

$$C(D_1 \vee D_2;x,y) = C(D_1;x,y)C(D_2;x,y).$$

Proof: Let $\mathfrak{C}(D)$ be the set of disjoint coverings of all vertices of D with directed paths and cycles. Each covering $(C_p(D), C_c(D)) \in \mathfrak{C}(D)$ is a pair of sets, where $C_p(D)$ is the set of all paths and $C_c(D)$ is the set of all cycles in this covering. Let $x, y \in \mathbb{N}$ be two natural numbers and D_1, D_2 be two disjoint digraphs. Define two sets $\mathscr{A}_{xy}$ and $\mathscr{B}_{xy}$ as

$$\begin{aligned}\mathscr{A}_{xy} := \{(f_1, g_1, f_2, g_2) \mid\ & f_1 : C_p(D_1) \to \{1, \ldots, x\} \text{ injective},\ g_1 : C_c(D_1) \to \{1, \ldots, y\},\\ & f_2 : C_p(D_2) \to \{1, \ldots, x\} \text{ injective},\ g_2 : C_c(D_2) \to \{1, \ldots, y\},\\ & (C_p(D_1), C_c(D_1)) \in \mathfrak{C}(D_1),\ (C_p(D_2), C_c(D_2)) \in \mathfrak{C}(D_2)\ \}\end{aligned}$$

$$\begin{aligned}\mathscr{B}_{xy} := \{(f, g) \mid\ & f : C_p(D_1 \vee D_2) \to \{1, \ldots, x\} \text{ injective},\ g : C_c(D_1 \vee D_2) \to \{1, \ldots, y\},\\ & (C_p(D_1 \vee D_2), C_c(D_1 \vee D_2)) \in \mathfrak{C}(D_1 \vee D_2)\ \}\end{aligned}$$

Because of the basic combinatorial counting argument, we know

$$|\mathscr{A}_{x,y}| = \left(\sum_{i,j} c_{i,j}(D_1) x^{\underline{i}} y^j\right)\left(\sum_{i,j} c_{i,j}(D_2) x^{\underline{i}} y^j\right) = C(D_1; x, y)C(D_2; x, y)$$

and

$$|\mathscr{B}_{x,y}| = \sum_{i,j} c_{i,j}(D_1 \vee D_2) x^{\underline{i}} y^j = C(D_1 \vee D_2; x, y).$$

In order to prove $C(D_1 \vee D_2; x, y) = C(D_1; x, y)C(D_2; x, y) \in \mathbb{Z}[x, y]$, we need to prove that for every $x, y \in \mathbb{N}$ the relation $|\mathscr{A}_{x,y}| = |\mathscr{B}_{x,y}|$ is valid.
Now we define a mapping $\Phi : \mathscr{A}_{x,y} \to \mathscr{B}_{x,y}$. Let $(C_p(D_1), C_c(D_1)) \in \mathfrak{C}(D_1)$ be a cycle-path covering of D_1, $(C_p(D_2), C_c(D_2)) \in \mathfrak{C}(D_2)$ be a path-cycle covering of D_2, $f_1 : C_p(D_1) \to \{1, \ldots, x\}$ and $f_2 : C_p(D_2) \to \{1, \ldots, x\}$ be injective functions mapping the paths into $\{1, \ldots, x\}$, and $g_1 : C_c(D_1) \to \{1, \ldots, y\}$, $g_2 : C_c(D_2) \to \{1, \ldots, y\}$ be functions mapping the cycles into $\{1, \ldots, y\}$. We construct a cycle-path covering $(C_p(D_1 \vee D_2), C_c(D_1 \vee D_2)) \in \mathfrak{C}(D_1 \vee D_2)$ of $D_1 \vee D_2$ and two functions $f : C_p(D_1 \vee D_2) \to \{1, \ldots, x\}$, $g : C_p(D_1 \vee D_2) \to \{1, \ldots, y\}$ such that $\Phi((f_1, g_1, f_2, g_2)) = (f, g)$. This covering contains all arcs of $(C_p(D_1), C_c(D_1))$ and $(C_p(D_2), C_c(D_2))$. If there are two paths $P \in C_p(D_1), P' \in C_p(D_2)$ such that $f_1(C_p(D_1)) = f_2(C_p(D_2))$, then we form a new path from P and P' by adding the arc from the last vertex of P to the first vertex of P' and let $f(P) := f_1(P) = f_2(P')$. Any other path or cycle in $(C_p(D_1 \vee D_2), C_c(D_1 \vee D_2))$ belongs either to $(C_p(D_1), C_c(D_1))$ or to $(C_p(D_2), C_c(D_2))$. They are mapped to the same value as what the same path or cycle is mapped to by f_1, f_2, g_1 and g_2. According to these rules, the paths in $C_p(D_1 \vee D_2)$ are mapped to different values by f, that is, f is injective.
Consider the function $\Phi^{-1} : \mathscr{B}_{x,y} \to \mathscr{A}_{x,y}$. For $f : C_p(D_1 \vee D_2) \to \{1, \ldots, x\}$ and $g : C_c(D_1 \vee D_2) \to \{1, \ldots, y\}$, $(C_p(D_1 \vee D_2), C_c(D_1 \vee D_2)) \in \mathfrak{C}(D_1 \vee D_2)$, let $\Phi^{-1}((f, g)) = (f_1, g_1, f_2, g_2)$, where g_1 and g_2 are restrictions of g to the set of cycles of $C_c(D_1 \vee D_2)$ lying in D_1 and D_2, respectively. f_1 and f_2 map the paths and parts of paths of $C_c(D_1 \vee D_2)$ lying in D_1 and D_2 to the numbers that the corresponding paths are mapped to by f,

respectively. The injectivity of f implies that f_1 and f_2 are injective.
Observe the functions Φ and Φ^{-1}, we can see $\Phi \circ \Phi^{-1} = \mathrm{id}_{\mathscr{B}_{x,y}}$ and $\Phi^{-1} \circ \Phi = \mathrm{id}_{\mathscr{A}_{x,y}}$. Therefore Φ^{-1} is the inverse function of Φ. That means, Φ is a bijection and thus $|\mathscr{A}_{x,y}| = |\mathscr{B}_{x,y}|$. This completes the proof. □

In [12], Chung and Graham generalized the cover polynomial and the geometric cover polynomial to the matrix cover polynomial (for matrices, that is, multidigraphs or weighted digraphs). In addition, a generalized cover polynomial $C_t(D;x,y)$ is defined using the same recurrence relation of the cover polynomial but the different initial condition

$$C_t(E_n;x,y) = x(x-t)\cdots(x-(n-1)t) = \prod_{i=0}^{n-1}(x-it).$$

Particularly, $C(D;x,y) = C_1(D;x,y)$ and $\widetilde{C}(D) = C_0(D;x,y)$.

3.3 Generalizations of Cycle and Path Polynomials for Digraphs

The next goal of this paper is to find the relationship between the (geometric) cover polynomial and our polynomials. Now we define the *bivariate cycle polynomial* $\widehat{\sigma}(D) = \widehat{\sigma}(D;x,y)$ and the *bivariate path polynomial* $\widehat{\pi}(D) = \widehat{\pi}(D;x,y)$ of a digraph D like the undirected graph version. Let $D = (V,E)$ be a digraph, let $kc(D)$ and $kp(D)$ denote the number of components of D which are directed cycles and directed paths, respectively. We define

$$\widehat{\sigma}(D) = \widehat{\sigma}(D;x,y) = \sum_F x^{|F|} y^{kc(D\langle F\rangle)},$$

where the sum is over all subsets F of E that each component of the spanning subgraph $D\langle F\rangle$ is either a directed cycle or an isolated vertex. And we define

$$\widehat{\pi}(D) = \widehat{\pi}(D;x,y) = \sum_F x^{|F|} y^{kp(D\langle F\rangle)},$$

where the sum is over all subsets F of E that each component of the spanning subgraph $D\langle F\rangle$ is either a directed path or an isolated vertex. Obviously $\widehat{\sigma}(D)$ and $\widehat{\pi}(D)$ are multiplicative under components, and $\widehat{\sigma}(E_n) = \widehat{\pi}(E_n) = 1$ for all $n \geq 0$. We have following recurrences for $\widehat{\sigma}(D)$ and $\widehat{\pi}(D)$:

Theorem 3.8

$$\widehat{\sigma}(D) = \begin{cases} \widehat{\sigma}(D_{-e}) + xy\widehat{\sigma}(D_{/e}) & \textit{if } e \textit{ is a loop,} \\ \widehat{\sigma}(D_{-e}) + x\widehat{\sigma}(D_{/e}) - x\widehat{\sigma}(D_{\dagger e}) & \textit{otherwise.} \end{cases}$$

$$\widehat{\pi}(D) = \begin{cases} \widehat{\pi}(D_{-e}) & \textit{if } e \textit{ is a loop,} \\ \widehat{\pi}(D_{-e}) + x\widehat{\pi}(D_{/e}) + x(y-1)\widehat{\pi}(D_{\dagger e}) & \textit{otherwise.} \end{cases}$$

Proof: Let $D = (V,E)$ be a digraph. For $\widehat{\sigma}(D)$, we enumerate the arc subsets $F \subseteq E$ such that each component of the spanning subgraph $D\langle F\rangle$ is either a directed cycle or an isolated vertex. For each $e \in E$ there are two kinds of F: either $e \notin F$ or $e \in F$.
If e is a loop in D, the arc subsets F of the first kind is counted by $\widehat{\sigma}(D_{-e})$. By the second kind, no other arcs in F can be incident to the loop e, and the rest of F corresponds to such an arc subset of $D_{\dagger e} = D_{/e}$. e contributes one cycle of length 1 and one arc to the polynomial. Thus, the second kind of F is enumerated by $xy\widehat{\sigma}(D_{/e})$.
If $e \in E$ is not a loop, the arc subsets F not containing e are counted by $\widehat{\sigma}(D_{-e})$. Consider now the digraph $D_{/e}$ and let w be the new resulting vertex after contraction. Since all arcs with the same head or the same tail as e are removed and the other arcs hold, each cycle of $D_{/e}$ containing w corresponds to a cycle of D containing e and vice versa. The cycles of $D_{/e}$ not containing w are identical to the cycles of $D_{\dagger e}$. However, e contributes one arc to the polynomial. Thus the subsets F of the second kind are enumerated by $x[\widehat{\sigma}(D_{/e}) - \widehat{\sigma}(D_{\dagger e})]$.

The recurrence relation for $\widehat{\sigma}(D)$ is obtained by summing up these cases.
Now consider $\widehat{\pi}(D)$. If e is a loop, the spanning subgraphs of D containing e do not contribute to the polynomial. The spanning subgraphs of D not containing e are the spanning subgraphs of D_{-e}. That is, $\widehat{\pi}(D) = \widehat{\pi}(D_{-e})$ if e is a loop.
If e is not a loop, in addition to the cases that contributed to the calculation of $\widehat{\sigma}(D)$ there is one more case: e is the only arc of a component of the spanning subgraph. Any arc incident to e cannot be in a spanning subgraph contributing to the polynomial, and e contributes one arc and one directed path to the polynomial. Thus the spanning subgraphs containing e as the only arc of a component, whose each component is either a directed path or an isolated vertex, are enumerated by $xy\widehat{\pi}(D_{\dagger e})$. Together with the other cases we obtain the recurrence relation. □

Furthermore, we can define the *trivariate cycle-path polynomial* $\widehat{\sigma\pi}(D) = \widehat{\sigma\pi}(D;x,y,z)$ of a digraph D counting all spanning subgraphs of D whose components are either directed cycles or directed paths or isolated vertices:

$$\widehat{\sigma\pi}(D;x,y,z) = \sum_{\substack{F\subseteq E\\ \forall v\in V:\deg^{+}_{D\langle F\rangle}(v)\le 1\\ \forall v\in V:\deg^{-}_{D\langle F\rangle}(v)\le 1}} x^{|F|}y^{kc(D\langle F\rangle)}z^{kp(D\langle F\rangle)}.$$

Because of the same arguments as in the proof of the last theorem, we have the following recurrence relation for $\widehat{\sigma\pi}(D)$:

Theorem 3.9 *$\widehat{\sigma\pi}(D) = \widehat{\sigma\pi}(D;x,y,z)$ satisfies the following recurrence relation*

$$\widehat{\sigma\pi}(D) = \begin{cases} \widehat{\sigma\pi}(D_{-e}) + xy\widehat{\sigma\pi}(D_{/e}) & \text{if } e \text{ is a loop,}\\ \widehat{\sigma\pi}(D_{-e}) + x\widehat{\sigma\pi}(D_{/e}) + x(z-1)\widehat{\sigma\pi}(D_{\dagger e}) & \text{otherwise.}\end{cases}$$

And the initial condition is $\widehat{\sigma\pi}(E_n) = 1$.

The following formulae follow direct from definition:

$$\sigma(D;x) = [y^1]\widehat{\sigma}(D;x,y),$$

$$\pi(D;x) = [y^1]\widehat{\pi}(D;x,y),$$

$$\widehat{\sigma}(D;x,y) = \widehat{\sigma\pi}(D;x,y,0),$$

$$\widehat{\pi}(D;x,y) = \widehat{\sigma\pi}(D;x,0,y).$$

The geometric cover polynomial counts the number of cycle-path covers of a digraph. Since isolated vertices are regarded as directed paths of length 0, the number of paths in a cycle-path cover equals the number of vertices minus the number of arcs in this cover. We have the following relationship.

Theorem 3.10 *If $D = (V,E)$ is a digraph, then*

$$\widetilde{C}(D;x,y) = x^{|V|}\widehat{\sigma\pi}(D;\frac{1}{x},y,1).$$

3.4 The Arc Elimination Polynomial for Digraphs

The digraph polynomials $C(D;x,y)$, $\sigma(D;x)$, $\pi(D;x)$, $\widehat{\sigma}(D;x,y)$, $\widehat{\pi}(D;x,y)$ and $\widehat{\sigma\pi}(D;x,y,z)$ satisfy certain linear recurrence relations with respect to deletion, contraction and extraction of an arc. In [5], Averbouch, Godlin and Makowsky introduced a most general undirected graph polynomial $\xi(G;x,y,z)$ satisfying an edge deletion-contraction-extraction linear recurrence relation, which generalizes the Tutte polynomial [33], the matching polynomial [16] and the bivariate chromatic polynomial [15]. The edge elimination polynomial is defined recursively as follows:

$$\begin{aligned}
&\xi(G;x,y,z) = \xi(G_{-e};x,y,z) + y\cdot\xi(G_{/e};x,y,z) + z\cdot\xi(G_{\dagger e};x,y,z),\\
&\xi(G_1\cup G_2;x,y,z) = \xi(G_1;x,y,z)\cdot\xi(G_2;x,y,z),\\
&\xi(E_1;x,y,z) = x,\\
&\xi(E_0;x,y,z) = 1.
\end{aligned}$$

In this section, we introduce the arc elimination polynomial for digraphs using the ideas of [4, 5].

Theorem 3.11 *The digraph polynomial $\widehat{\xi}(D) = \widehat{\xi}(D;t,x,y,z)$ satisfying the recurrence relation*

$$\begin{aligned}
&\widehat{\xi}(D;t,x,y,z) = t\cdot\widehat{\xi}(D_{-e};t,x,y,z) + y\cdot\widehat{\xi}(D_{/e};t,x,y,z) + z\cdot\widehat{\xi}(D_{\dagger e};t,x,y,z),\\
&\widehat{\xi}(G_1\cup G_2;x,y,z) = \widehat{\xi}(G_1;t,x,y,z)\cdot\widehat{\xi}(G_2;t,x,y,z),\\
&\widehat{\xi}(E_1;t,x,y,z) = x,\\
&\widehat{\xi}(E_0;t,x,y,z) = 1
\end{aligned}$$

is well-defined iff $t=1$ or $y=z=0$. In the latter case, $\widehat{\xi}(D) = t^{|E(D)|}x^{|V(D)|}$.

Proof: First, we prove that $t=1$ or $y=z=0$ is the necessary condition for the well-definedness of $\widehat{\xi}(D)$. First consider two arcs $e=(u,v)$, $f=(v,w)$ in $E(D)$, where u, v and w are different vertices. In order to be well-defined, $\widehat{\xi}(D)$ must return the same value when the decomposition is applied first to the arc e and then to the arc f, as well as when it is applied first to f then to e.
Applying decomposition first to e then to f, we have

$$\begin{aligned}
\widehat{\xi}(D) &= t\cdot\widehat{\xi}(D_{-e}) + y\cdot\widehat{\xi}(D_{/e}) + z\cdot\widehat{\xi}(D_{\dagger e})\\
&= t^2\cdot\widehat{\xi}(D_{-e-f}) + ty\cdot\widehat{\xi}(D_{-e/f}) + tz\cdot\widehat{\xi}(D_{-e\dagger f})\\
&\quad + ty\cdot\widehat{\xi}(D_{/e-f}) + y^2\cdot\widehat{\xi}(D_{/e/f}) + yz\cdot\widehat{\xi}(D_{/e\dagger f}) + z\cdot\widehat{\xi}(D_{\dagger e})\\
&= t^2\cdot\widehat{\xi}(D_{-e-f}) + ty\cdot\widehat{\xi}(D_{-e/f}) + tz\cdot\widehat{\xi}(D_{\dagger f})\\
&\quad + ty\cdot\widehat{\xi}(D_{-f/e}) + y^2\cdot\widehat{\xi}(D_{/e/f}) + yz\cdot\widehat{\xi}(D_{\dagger e\dagger f}) + z\cdot\widehat{\xi}(D_{\dagger e}),
\end{aligned}$$

and first on f then on e, we have

$$\begin{aligned}\widehat{\xi}(D) &= t\cdot\widehat{\xi}(D_{-f})+y\cdot\widehat{\xi}(D_{/f})+z\cdot\widehat{\xi}(D_{\dagger f})\\ &= t^2\cdot\widehat{\xi}(D_{-f-e})+ty\cdot\widehat{\xi}(D_{-f/e})+tz\cdot\widehat{\xi}(D_{-f\dagger e})\\ &\quad +ty\cdot\widehat{\xi}(D_{/f-e})+y^2\cdot\widehat{\xi}(D_{/f/e})+yz\cdot\widehat{\xi}(D_{/f\dagger e})+z\cdot\widehat{\xi}(D_{\dagger f})\\ &= t^2\cdot\widehat{\xi}(D_{-e-f})+ty\cdot\widehat{\xi}(D_{-f/e})+tz\cdot\widehat{\xi}(D_{\dagger e})\\ &\quad +ty\cdot\widehat{\xi}(D_{-e/f})+y^2\cdot\widehat{\xi}(D_{/e/f})+yz\cdot\widehat{\xi}(D_{\dagger e\dagger f})+z\cdot\widehat{\xi}(D_{\dagger f}).\end{aligned}$$

They must coincide because of the well-definedness of $\widehat{\xi}(D)$. We have

$$tz\cdot\widehat{\xi}(D_{\dagger f})+z\cdot\widehat{\xi}(D_{\dagger e})=tz\cdot\widehat{\xi}(D_{\dagger e})+z\cdot\widehat{\xi}(D_{\dagger f}),$$

that is,

$$(t-1)z\cdot\widehat{\xi}(D_{\dagger e})=(t-1)z\cdot\widehat{\xi}(D_{\dagger f}),$$

which leads to $t=1$ or $z=0$ or $\widehat{\xi}(D_{\dagger e})=\widehat{\xi}(D_{\dagger f})$.
Consider the latter case. Let D be a digraph and v an arbitrary vertex of D. Let D' be the digraph obtained from D by adding two vertices $u,w\notin V(D)$ and two arcs $e=(v,u), f=(u,w)$ to D. Applying extraction on e and f, we have $D'_{\dagger e}=D_{-v}\cup K_1$ and $D'_{\dagger f}=D$. Since $\widehat{\xi}(D'_{\dagger e})=\widehat{\xi}(D'_{\dagger f})$, we have $\widehat{\xi}(D_{-v}\cup E_1)=\widehat{\xi}(D)$ for any vertices $v\in V(D)$. Applying this on every vertex of D, we get a trivial polynomial $\widehat{\xi}(D)=\widehat{\xi}(E_{|V(D)|})=x^{|V(D)|}$. This is a evaluation of $\widehat{\xi}(D)$ at $t=1,\ y=z=0$. That is, the third case is contained in the first case.
Consider now the second case $\widehat{\xi}(D;t,x,y,0)$ and two arcs $e=(u,v)$, $f=(w,v)$ in $E(D)$, where u, v and w are different. Applying decomposition first on e then on f we get

$$\begin{aligned}\widehat{\xi}(D;t,x,y,0) &= t\cdot\widehat{\xi}(D_{-e};t,x,y,0)+y\cdot\widehat{\xi}(D_{/e};t,x,y,0)\\ &= t^2\cdot\widehat{\xi}(D_{-e-f};t,x,y,0)+ty\cdot\widehat{\xi}(D_{-e/f};t,x,y,0)+y\cdot\widehat{\xi}(D_{/e};t,x,y,0)\\ &= t^2\cdot\widehat{\xi}(D_{-e-f};t,x,y,0)+ty\cdot\widehat{\xi}(D_{/f};t,x,y,0)+y\cdot\widehat{\xi}(D_{/e};t,x,y,0)\end{aligned}$$

Applying decomposition first on f then on e, we get

$$\begin{aligned}\widehat{\xi}(D;t,x,y,0) &= t\cdot\widehat{\xi}(D_{-f};t,x,y,0)+y\cdot\widehat{\xi}(D_{/f};t,x,y,0)\\ &= t^2\cdot\widehat{\xi}(D_{-f-e};t,x,y,0)+ty\cdot\widehat{\xi}(D_{-f/e};t,x,y,0)+y\cdot\widehat{\xi}(D_{/f};t,x,y,0)\\ &= t^2\cdot\widehat{\xi}(D_{-e-f};t,x,y,0)+ty\cdot\widehat{\xi}(D_{/e};t,x,y,0)+y\cdot\widehat{\xi}(D_{/f};t,x,y,0).\end{aligned}$$

From the coincidence of two results we have

$$(t-1)y\cdot\widehat{\xi}(D_{/e};t,x,y,0)=(t-1)y\cdot\widehat{\xi}(D_{/f};t,x,y,0).$$

The well-definedness implies that $t=1$ or $y=0$ or $\widehat{\xi}(D_{/e})=\widehat{\xi}(D_{/f})$. If $y=0$, then $\widehat{\xi}(D)=t\cdot\widehat{\xi}(D_{-e})$ and $\widehat{\xi}(E_n)=x^n$, which yields immediately that $\widehat{\xi}(D)=t^{|E(D)|}x^{|V(D)|}$.

If $\widehat{\xi}(D_{/e}) = \widehat{\xi}(D_{/f})$, given any digraph D and let v be any vertex of D. Let D' be the digraph obtained from D by adding two vertices $u, w \notin V(D)$ and two arcs $e = (v, u)$, $f = (w, u)$ to D. Applying the contraction on e and f, we have $D'/f = D \cup E_1$ and $D'/f = D_{-E^+(v)} \cup E_1$. $\widehat{\xi}(D'_{/e}) = \widehat{\xi}(D'_{/f})$ implies

$$\widehat{\xi}(D \cup E_1) = \widehat{\xi}(D_{-E^+(v)} \cup E_1).$$

From the definition of $\widehat{\xi}(D)$ we have

$$\widehat{\xi}(D) = \widehat{\xi}(D_{-E^+(v)})$$

for any digraph D and any vertex v in D. Applying $D_{-E^+(v)}$ on every vertex of D, we have $\widehat{\xi}(D) = \widehat{\xi}(E_{|V(D)|}) = x^{|V(D)|}$. In this case, it is the trivial polynomial $\widehat{\xi}(D; 1, x, 0, 0) = x^{|V(D)|}$.

So far, we proved that the necessary condition is $t = 1$ or $y = z = 0$. The well-definedness in case $y = z = 0$ is ensured by the explicit formula $\widehat{\xi}(D) = t^{|E(D)|} x^{|V(D)|}$. Consider the case $t = 1$. We denote this possible polynomial by the notation of edge elimination polynomial:

$$\xi(D; x, y, z) := \widehat{\xi}(D; 1, x, y, z).$$

Then we should prove the well-definedness of $\xi(D; x, y, z)$, that is, the result is independent of the order of decomposition steps.

The distributivity of multiplication implies that elimination of an arc is exchangeable with decomposition of disjoint union. Hence, we can assume that the disjoint union decomposition steps are applied only on empty graphs, and only consider the order of decomposition of arcs.

We shall consider only the linear order over arcs rather than decomposition steps. Such an order uniquely determines the decomposition process, if by convention, we just skip the steps of removing arcs that have been already removed by the proceeding steps. It is enough to show that successively decomposed arcs can be swapped. For two arcs $e, f \in E(D)$ there are 11 possible cases as shown in Figure 3.4. In the case 1-3, the arc elimination operations are independent and hence commutative. In case 4 and case 5 the exchangeablility of elimination order of e and f are already showed. The case 6 is the same as case 5. In the case 7 and 8 we decompose first on e then on f and have

$$\begin{aligned}
\xi(D; x, y, z) &= \xi(D_{-e}; x, y, z) + y \cdot \xi(D_{/e}; x, y, z) + z \cdot \xi(D_{\dagger e}; x, y, z) \\
&= \xi(D_{-e-f}; x, y, z) + y \cdot \xi(D_{-e/f}; x, y, z) + z \cdot \xi(D_{-e\dagger f}; x, y, z) \\
&\quad + y \cdot \xi(D_{/e}; x, y, z) + z \cdot \xi(D_{\dagger e}; x, y, z) \\
&= \xi(D_{-e-f}; x, y, z) + (y+z) \cdot \xi(D_{\dagger f}; x, y, z) + y \cdot \xi(D_{/e}; x, y, z) + z \cdot \xi(D_{\dagger e}; x, y, z).
\end{aligned}$$

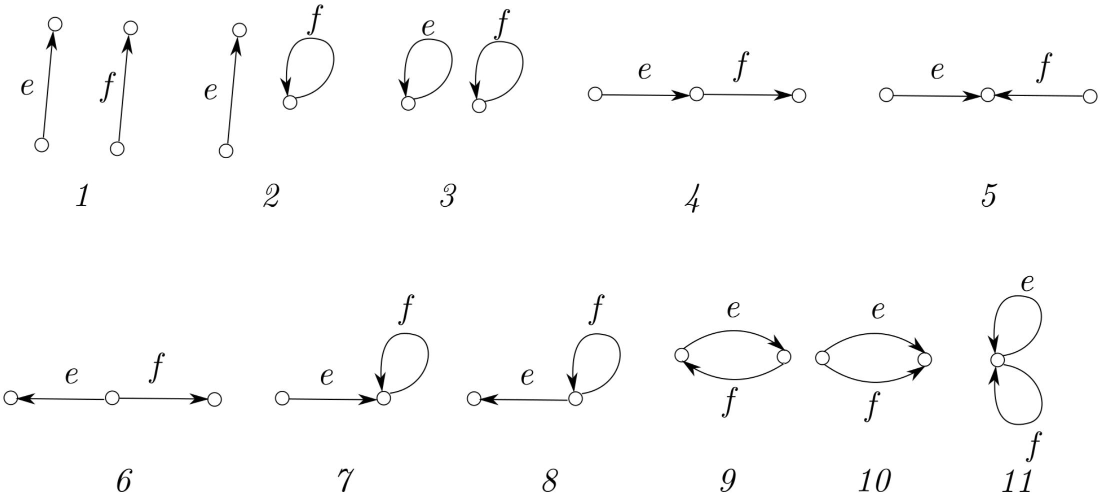


Figure 3.4: 11 cases for two arcs e and f

Applying decomposition first on f then on e, we get

$$\begin{aligned}
\xi(D;x,y,z) &= \xi(D_{-f};x,y,z) + y\cdot\xi(D_{/f};x,y,z) + z\cdot\xi(D_{\dagger f};x,y,z)\\
&= \xi(D_{-f-e};x,y,z) + y\cdot\xi(D_{-f/e};x,y,z) + z\cdot\xi(D_{-f\dagger e};x,y,z)\\
&\quad + y\cdot\xi(D_{/f};x,y,z) + z\cdot\xi(D_{\dagger f};x,y,z)\\
&= \xi(D_{-e-f};x,y,z) + y\cdot\xi(D_{/e};x,y,z) + z\cdot\xi(D_{\dagger e};x,y,z) + (y+z)\cdot\xi(D_{\dagger f};x,y,z).
\end{aligned}$$

These two expressions are equal.
We check the case 9 similarly:

$$\begin{aligned}
\xi(D;x,y,z) &= \xi(D_{-e};x,y,z) + y\cdot\xi(D_{/e};x,y,z) + z\cdot\xi(D_{\dagger e};x,y,z)\\
&= \xi(D_{-e-f};x,y,z) + y\cdot\xi(D_{-e/f};x,y,z) + z\cdot\xi(D_{-e\dagger f};x,y,z) + y\cdot\xi(D_{/e-f};x,y,z)\\
&\quad + y^2\cdot\xi(D_{/e/f};x,y,z) + yz\cdot\xi(D_{/e\dagger f};x,y,z) + z\cdot\xi(D_{\dagger e};x,y,z)\\
&= \xi(D_{-e-f};x,y,z) + y\cdot\xi(D_{-e/f};x,y,z) + y\cdot\xi(D_{-f/e};x,y,z)\\
&\quad + (y^2+yz+2z)\cdot\xi(D_{\dagger e};x,y,z),
\end{aligned}$$

and

$$\begin{aligned}
\xi(D;x,y,z) &= \xi(D_{-f};x,y,z) + y\cdot\xi(D_{/f};x,y,z) + z\cdot\xi(D_{\dagger f};x,y,z)\\
&= \xi(D_{-f-e};x,y,z) + y\cdot\xi(D_{-f/e};x,y,z) + z\cdot\xi(D_{-f\dagger e};x,y,z) + y\cdot\xi(D_{/f-e};x,y,z)\\
&\quad + y^2\cdot\xi(D_{/f/e};x,y,z) + yz\cdot\xi(D_{/f\dagger e};x,y,z) + z\cdot\xi(D_{\dagger f};x,y,z)\\
&= \xi(D_{-e-f};x,y,z) + y\cdot\xi(D_{-e/f};x,y,z) + y\cdot\xi(D_{-f/e};x,y,z)\\
&\quad + (y^2+yz+2z)\cdot\xi(D_{\dagger e};x,y,z),
\end{aligned}$$

we have the same result.
In the case 10 and 11, the arc elimination steps are symmetric in their transformations of D with respect to the order among e and f. We have analyzed all of the cases and these complete the proof. □

Definition 3.12 The *arc elimination polynomial* of a digraph D is defined recursively as follows:

$$\begin{aligned}
&\xi(D;x,y,z) = \xi(D_{-e};x,y,z) + y\cdot\xi(G_{/e};x,y,z) + z\cdot\xi(G_{\dagger e};x,y,z) \quad \forall e\in E(D),\\
&\xi(G_1\cup G_2;x,y,z) = \xi(G_1;x,y,z)\cdot\xi(G_2;x,y,z),\\
&\xi(E_1;x,y,z) = x,\\
&\xi(E_0;x,y,z) = 1.
\end{aligned}$$

The recurrence relation of the trivariate cycle-path polynomial contains a case distinction. Motivated by the co-reduction of the Tutte polynomial

$$\begin{aligned}
T(G;x,y) &= \sum_{F\subseteq E(G)} (x-1)^{k(G\langle F\rangle)-k(G)}(y-1)^{|F|+k(G\langle F\rangle)-|V(G)|}\\
&= \begin{cases} 1 & \text{if } G \text{ has no edges},\\ xT(G_{-e};x,y) & \text{if } e \text{ is a bridge},\\ yT(G_{/e};x,y) & \text{if } e \text{ is a loop},\\ T(G_{-e};x,y)+T(G_{/e};x,y) & \text{otherwise} \end{cases}
\end{aligned}$$

and the dichromatic polynomial

$$Z(G;q,v) = \sum_{F\subseteq E(G)} q^{k(G\langle F\rangle)}v^{|F|} = \begin{cases} q^{|V(G)|} & \text{if } G \text{ has no edges},\\ Z(G_{-e};q,v)+vZ(G_{/e};q,v) & \text{for an edge } e \end{cases}$$

by

$$\begin{aligned}
T(G;x,y) &= (x-1)^{-k(G)}(y-1)^{-|V(G)|}Z(G;(x-1)(y-1),y-1),\\
Z(G;q,v) &= \left(\frac{q}{v}\right)^{k(G)} v^{|V(G)|}T(G;\frac{q}{v}+1,v+1),
\end{aligned}$$

we pose a question: can we introduce a variable for the initial condition in order to avoid the case distinction, that is, can $\xi(D;x,y,z)$ be determined by $\widehat{\sigma\pi}(D;x,y,z)$ and vice versa? The answer is positive, since the number of vertices after the decomposition contains information about how many arc extraction operations are applied on the loops.

Theorem 3.13 *The arc elimination polynomial $\xi(D;x,y,z)$ and the trivariate cycle-path polynomial $\widehat{\sigma\pi}(D;x,y,z)$ are co-reducible via*

$$\widehat{\sigma\pi}(D;x,y,z) = \left(\frac{y-1}{z-1}\right)^{|V(D)|}\xi\left(D;\frac{y-1}{z-1},x\frac{y-1}{z-1},x\frac{(y-1)^2}{z-1}\right)$$

and

$$\xi(D;x,y,z) = x^{|V(D)|}\widehat{\sigma\pi}\left(D;\frac{y}{x},\frac{y+z}{y},\frac{z}{xy}+1\right).$$

Proof: We consider only the arc elimination of a digraph D into empty graphs (at last the disjoint union decomposition may be applied). The result M is a multiset of empty digraphs over $\{E_0,\ldots,E_{|V(D)|}\}$. Since $\xi(D;x,y,z)$ and $\widehat{\sigma\pi}(D;x,y,z)$ are well-defined, the multiset M is independent of the order of arc decomposition. Choose a fixed but arbitrary order of decomposition of D on the arcs into the multiset of empty digraphs M. For each $m\in M$, we denote the number of contraction steps on the loops resulting m in this decomposition by $a_l(m)$. Similarly, we denote the number of contraction steps on the non-loop arcs, the number of extraction steps on the loops and non-loop arcs resulting m by $a_2(m)$, $b_1(m)$ and $b_2(m)$, respectively.
Then from the recurrence relation

$$\xi(D;x,y,z)=\xi(D_{-e};x,y,z)+y\cdot\xi(G_{/e};x,y,z)+z\cdot\xi(G_{\dagger e};x,y,z)$$

we have the following expression of $\xi(D;q,v,w)$:

$$\xi(D;q,v,w)=\sum_{m\in M}q^{|V(m)|}v^{a_1(m)+a_2(m)}w^{b_1(m)+b_2(m)}.$$

Since the arc deletion operation has no influence on the vertices, the arc contraction and loop extraction remove one vertex and extraction of a non-loop arc removes two vertices, we have $|V(m)|=|V(D)|-a_1(m)-a_2(m)-b_1(m)-2b_2(m)$ and hence

$$\xi(D;q,v,w)=q^{|V(D)|}\sum_{m\in M}q^{-a_1(m)-a_2(m)-b_1(m)-2b_2(m)}v^{a_1(m)+a_2(m)}w^{b_1(m)+b_2(m)}.$$

Recall that the recurrence relation of $\widehat{\sigma\pi}(D;x,y,z)$ is

$$\widehat{\sigma\pi}(D;x,y,z)=\begin{cases}\widehat{\sigma\pi}(D_{-e};x,y,z)+xy\widehat{\sigma\pi}(D_{/e};x,y,z) & e\text{ is a loop,}\\ \widehat{\sigma\pi}(D_{-e};x,y,z)+x\widehat{\sigma\pi}(D_{/e};x,y,z)+x(z-1)\widehat{\sigma\pi}(D_{\dagger e};x,y,z) & \text{otherwise.}\end{cases}$$

Since $D_{/e}=D_{\dagger e}$ if e is a loop, we may say

$$\widehat{\sigma\pi}(D;x,y,z)=\widehat{\sigma\pi}(D_{-e};x,y,z)+(xy-\alpha)\widehat{\sigma\pi}(D_{/e};x,y,z)+\alpha\widehat{\sigma\pi}(D_{\dagger e};x,y,z),$$

if e is a loop, where α can be chosen arbitrarily. Then we have the following expressions of $\widehat{\sigma\pi}(D)$:

$$\widehat{\sigma\pi}(D;x,y,z)=\sum_{m\in M}(xy-\alpha)^{a_1(m)}\alpha^{b_1(m)}x^{a_2(m)}(x(z-1))^{b_2(m)}.$$

Setting $\alpha=xy-x$, we get

$$\widehat{\sigma\pi}(D;x,y,z)=\sum_{m\in M}(xy-x)^{b_1(m)}x^{a_1(m)+a_2(m)}(x(z-1))^{b_2(m)}.$$

Equation $\xi(D;q,v,w) = q^{|V(D)|}\widehat{\sigma\pi}(D;x,y,z)$ holds, if

$$q^{-a_1(m)-a_2(m)-b_1(m)-2b_2(m)}v^{a_1(m)+a_2(m)}w^{b_1(m)+b_2(m)} = (xy-x)^{b_1(m)}x^{a_1(m)+a_2(m)}(x(z-1))^{b_2(m)},$$

that is,

$$\left(\frac{v}{q}\right)^{a_1(m)+a_2(m)}\left(\frac{w}{q^2}\right)^{b_1(m)+b_2(m)}q^{b_1(m)} = x^{a_1(m)+a_2(m)}(x(z-1))^{b_1(m)+b_2(m)}\left(\frac{y-1}{z-1}\right)^{b_1(m)}.$$

Applying "equating exponents", we conclude that

$$x = \frac{v}{q},\ y = \frac{v+w}{v},\ z = \frac{w}{qv}+1$$

or

$$q = \frac{y-1}{z-1},\ v = x\frac{y-1}{z-1},\ w = x\frac{(y-1)^2}{z-1},$$

this completes the proof. □

We have now an interest in the combinatorial interpretation of the coefficients of $\xi(D;x,y,z)$. In the next theorem, an explicit expression of the arc elimination polynomial is given.

Theorem 3.14

$$\xi(D;x,y,z) = \sum_{A,B} x^{k(D\langle A\cup B\rangle)-c(D\langle B\rangle)-c_1(D\langle A\rangle)}y^{|A|+|B|-c(D\langle B\rangle)}z^{c(D\langle B\rangle)},$$

where the sum is over all subsets $A,B\subseteq E(D)$ of $E(D)$ such that

1. *$A\cap B=\emptyset$,*
2. *there is no vertex such that an arc in A and an arc in B are incident to it, and*
3. *each component of the spanning subgraph $D\langle A\cup B\rangle$ is either a cycle or a path or an isolated vertex.*

Here $k(D)$ denotes the number of components of D, $c(D)$ denotes the number of covered components of D, that is, components of D which are not isolated vertices, and $c_1(D)$ denotes the number of cycles of length 1 (loops) in D.

Proof: Let $D=(V,E)$ be a (multi-)digraph. The set of pairs (A,B) of arc subsets $A,B\subseteq E$ satisfying the three conditions in the theorem is denoted by $\mathscr{C}(D)$. Let $N(D)$ be defined explicitly as

$$N(D;x,y,z) := \sum_{(A,B)\in\mathscr{C}(D)} x^{k(D\langle A\cup B\rangle)-c(D\langle B\rangle)-c_1(D\langle A\rangle)}y^{|A|+|B|-c(D\langle B\rangle)}z^{c(D\langle B\rangle)}.$$

We may use the notation $f(D,(A,B)) := x^{k(D\langle A\cup B\rangle)-c(D\langle B\rangle)-c_1(D\langle A\rangle)}y^{|A|+|B|-c(D\langle B\rangle)}z^{c(D\langle B\rangle)}$, then $N(D;x,y,z) := \sum_{(A,B)\in\mathscr{C}(D)} f(D,(A,B))$.

In order to proof $\xi(D;x,y,z) = N(D;x,y,z)$, we need to show that $N(D)$ satisfies

$$N(D;x,y,z) = N(D_{-e};x,y,z) + y \cdot N(G_{/e};x,y,z) + z \cdot N(G_{\dagger e};x,y,z) \quad \forall e \in E,$$
$$N(E_n;x,y,z) = x^n.$$

For the empty digraph E_n, the only summand corresponds to $A = B = \emptyset$, and obviously $N(E_n;x,y,z) = x^n = \xi(E_n;x,y,z)$.
Let $e \in E$ be an arbitrarily chosen arc. The summands can be divided into three disjoint cases:

- Case 1: $e \notin A \cup B$;
- Case 2: $e \in B$ and e is the only arc of a component of $D\langle B\rangle$;
- Case 3: All the rest. That is, $e \in A$ or $e \in B$ but it is not the only arc of a component of $D\langle B\rangle$.

The sets of arc subset pairs $(A,B) \in \mathscr{C}(D)$ satisfying the conditions in case 1, 2 and 3 are denoted by $\mathscr{C}_1(D)$, $\mathscr{C}_2(D)$ and $\mathscr{C}_3(D)$, respectively.
In the case 1, it is easily to seen that $\mathscr{C}_1(D) = \mathscr{C}(D_{-e})$. Then

$$N_1(D) := \sum_{(A,B)\in\mathscr{C}_1(D)} x^{k(D\langle A\cup B\rangle)-c(D\langle B\rangle)-c_1(D\langle A\rangle)} y^{|A|+|B|-c(D\langle B\rangle)} z^{c(D\langle B\rangle)} = N(D_{-e}).$$

In the case 2, $e \in B$ is the only arc of a component of $D\langle B\rangle$, because of the required condition, any arc incident to e can not in A or B. Thus we can define a bijection $\varphi : \mathscr{C}_2(D) \to \mathscr{C}(D_{\dagger e})$, $\varphi((A,B)) := (A, B\backslash\{e\})$. Now compare $D_{\dagger e}$ with D, we get

$$\begin{aligned}
&|B\backslash\{e\}| = |B| - 1,\\
&k(D_{\dagger e}\langle A \cup B\backslash\{e\}\rangle) = k(D\langle A\cup B\rangle) - 1, \text{ and}\\
&c(D_{\dagger e}\langle B\backslash\{e\}\rangle) = c(D\langle B\rangle) - 1.
\end{aligned}$$

that is,

$$f(D,(A,B)) = z \cdot f(D_{\dagger e}, \varphi((A,B))) \quad \forall (A,B) \in \mathscr{C}_2(D)$$

and therefore,

$$\begin{aligned}
N_2(D) :=& \sum_{(A,B)\in\mathscr{C}_2(D)} x^{k(D\langle A\cup B\rangle)-c(D\langle B\rangle)-c_1(D\langle A\rangle)} y^{|A|+|B|-c(D\langle B\rangle)} z^{c(D\langle B\rangle)}\\
=& \sum_{(A,B)\in\mathscr{C}_2(D)} f(D,(A,B))\\
=& \; z \cdot \sum_{(A,B)\in\mathscr{C}_2(D)} f(D_{\dagger e}, \varphi((A,B)))\\
=& \; z \cdot \sum_{(A,B)\in\mathscr{C}(D_{\dagger e})} f(D_{\dagger e},(A,B))\\
=& \; z \cdot N(D_{\dagger e}).
\end{aligned}$$

In the case 3, either $e \in A$ or $e \in B$ and e is incident to other arcs in B. Since e is either the only arc of a component of $D\langle A\rangle$, or belongs to a directed path or a directed cycle of length at least two, whose arcs are either all in A or all in B, we can define a function $\psi : \mathscr{C}_3(D) \to \mathscr{C}(D_{/e})$, $\psi((A,B)) := (A\backslash\{e\}, B\backslash\{e\})$. Evidently

$$\psi^{-1}((A,B)) := \begin{cases} (A, B\cup\{e\}) & \text{if } e \text{ is incident to an arc of } B, \\ (A\cup\{e\}, B) & \text{otherwise} \end{cases}$$

is the inverse function of ψ, and the well-definedness of ψ^{-1} is guaranteed by the conditions of (A,B), we conclude that ψ is bijective. Compare now $D_{/e}$ with D, we get

$$\begin{aligned} &|A\backslash\{e\}| + |B\backslash\{e\}| = |A| + |B| - 1, \\ &c(D_{/e}\langle B\backslash\{e\}\rangle) = c(D\langle B\rangle), \\ &k(D_{/e}\langle A\cup B\backslash\{e\}\rangle) = \begin{cases} k(D\langle A\cup B\rangle) - 1 & \text{if } e\in A \text{ is a loop}, \\ k(D\langle A\cup B\rangle) & \text{otherwise}, \end{cases} \\ &c_1(D_{/e}\langle A\rangle) = \begin{cases} c_1(D\langle A\rangle) - 1 & \text{if } e\in A \text{ is a loop}, \\ c_1(D\langle A\rangle) & \text{otherwise}. \end{cases} \end{aligned}$$

Applying to the function f, we have

$$f(D,(A,B)) = y\cdot f(D_{/e}, \psi((A,B))) \quad \forall (A,B)\in\mathscr{C}_3(D).$$

Therefore,

$$\begin{aligned} N_3(D) &:= \sum_{(A,B)\in\mathscr{C}_3(D)} x^{k(D\langle A\cup B\rangle)-c(D\langle B\rangle)-c_1(D\langle A\rangle)} y^{|A|+|B|-c(D\langle B\rangle)} z^{c(D\langle B\rangle)} \\ &= \sum_{(A,B)\in\mathscr{C}_3(D)} f(D,(A,B)) \\ &= y\cdot \sum_{(A,B)\in\mathscr{C}_3(D)} f(D_{\dagger e}, \psi((A,B))) \\ &= y\cdot \sum_{(A,B)\in\mathscr{C}(D_{/e})} f(D_{/e},(A,B)) \\ &= y\cdot N(D_{/e}). \end{aligned}$$

Summing up the three cases, we conclude that

$$N(D) = N_1(D) + N_2(D) + N_3(D) = N(D_{-e}) + y\cdot N(D_{/e}) + z\cdot N(D_{\dagger e}).$$

Together with $N(E_n) = x^n$ it implies $N(D) = \xi(D)$. This completes the proof. $\square$

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Erklärung

Hiermit erkläre ich, dass ich meine Arbeit selbstständig verfasst, keine anderen als die angegebenen Quellen und Hilfsmittel benutzt und die Arbeit noch nicht anderweitig für Prüfungszwecke vorgelegt habe.

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Mittweida, 25. Juli 2017